Gauge Theories in Three Dimensions

by

Anthony Brian Waites, B.Sc.(Hons.)

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Declaration

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Anthony B. Waites

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Abstract

Field theories in 2+1 space-time dimensions are of interest both intrinsically, due to their novel properties such as actions which are topologically non-trivial, and also due to their ability to explain of phenomena such as the fractional quantum Hall effect and certain behaviour of high T_c superconductors, and for their use in conformal field theory in 2D.

This thesis begins by considering scalar and spinor QED in 2+1 dimensions, performing perturbation theory to study its behaviour (without allowing the presence or dynamical generation of a parity-violating photon mass). It is found, as first noted by Jackiw and Templeton, that an IR instability prohibits such a perturbative study. The gauge technique is adopted as a non-perturbative alternative, and the photon is allowed to be "dressed" in a cloud of fermion loops, yielding results which encompass the perturbation results in the UV region, whilst remaining finite at IR momenta.

Chern-Simons theory is then considered, where the photon is allowed to acquire a parity-violating mass. In order to use dimensional regularization to handle the apparently UV divergent integrals which appear, a new formulation of the theory is proposed, allowing the action to be written in arbitrary D dimensions, so that the integrals can be safely evaluated. It is also found that the IR problems which plague the conventional theory are no longer present, as the photon propagator behaviour has been "softened" by the photon mass, allowing perturbation results to be obtained.

Finally, the idea of mass generation within these theories is considered in more detail, where we see that the presence of a fermion mass will cause a photon mass to be dynamically generated, and vice versa. These ideas are then generalized for arbitrary odd dimensional parity-violating theories.

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“As far as we can discern, the sole purpose of human existence is to kindle a light in the darkness of mere being.”
— Carl Jung

“The proof that the little prince existed is that he was charming, that he laughed, and that he was looking for a sheep. If anybody wants a sheep, that is a proof that he exists.”
— *The Little Prince*, Antoine de Saint-Exupéry

“Everything that happens once can never happen again. But everything that happens twice will surely happen a third time.”
— *Proverb*

“‘Yes,’ said the ferryman, ‘it is a very beautiful river. I love it above everything. I have often listened to it, gazed at it, and I have always learned something from it. One can learn much from a river.’”
— *Siddhartha*, Hermann Hesse

“There’s more to you young Haroun Khalifa, than meets the blinking eye.”
— *Haroun and the sea of stories*, Salman Rushdie

“The most wasted of all days is that on which one has not laughed.”
— Nicolas Chamfort

"...the river is everywhere at the same time, at the source and at the mouth, at the waterfall, at the ferry, at the current, in the ocean and in the mountains, everywhere, and ...the present only exists for it, not the shadow of the past, nor the shadow of the future"
— *Siddhartha*, Hermann Hesse

"The best way to know God is to love many things."
— Vincent van Gogh

"That's right. When I was your age, television was called books."
— *The grandfather* in *'The Princess Bride'*

"It seems to me, Govinda, that love is the most important thing in the world. It may be important to great thinkers to examine the world, to explain and despise it. But I think it is only important to love the world, not to despise it, not for us to hate each other, but to be able to regard the world and ourselves and all beings with love, admiration and respect."
— *Siddhartha*, Hermann Hesse

"He didn't fall? INCONCEIVABLE!"
"You keep using that word. I do not think it means what you think it means."
— *Vizzini and Inigo* in *'The Princess Bride'*

"Frank Burns eats worms"
— *Hawkeye Pierce* in *M*A*S*H*

Chapter 1

Introduction

The purpose of this introductory chapter is to place the subject matter of this thesis within a historical perspective. We begin by outlining the progress made in the analysis of field theory in three dimensions, then give a review of the gauge technique, the non-perturbative technique we will exploit where necessary in our calculations. The structure of the thesis is outlined in the final section.

1.1 Field Theory in (2+1)D

When studying gauge theories, it seems natural to look at a theory set in $3+1$ space-time dimensions, as the physical world is set within such a geometry. Extensive research has been conducted on such theories, with considerable success. Quantum electrodynamics is the simplest gauge theory to be physically meaningful, describing the quantized interactions of photons and electrons. It is an abelian theory, being described by the group $U(1)$, and was found to be renormalizable [1–3], requiring only two renormalization constants [4]. In the non-abelian case, the electro-weak or $SU(2) \times U(1)$ gauge theory [5,6] together with spontaneous symmetry breaking [7–9], unifies the electromagnetic and weak interactions, and also places a self-consistent theoretical framework around all of the phenomenological weak models. Renormalization has permitted the terms in the perturbation expansion to be rendered finite [10], and the identification of the intermediate vector bosons [11–13] has given the theory its necessary verification. Another successful theory in (3+1)D is quantum chromodynamics (QCD) [14–17]. QCD is the gauge theory of the $SU(3)$ colour group, and is largely accepted as the theory describing the strong interaction. It provides a theoretical foundation for the quark model [18–21] and can be used to explain the results of deep inelastic scattering [22–24]. It is not considered as successful as the above theories as it has so far been unable to supply a convincing explanation of confinement, which is the process preventing the detection of single quarks or coloured particles. It is possible that some insight may be gained by considering a theory set in $2+1$ dimensions. As well as having intrinsically interesting features, it is thought that a (2+1)D theory could be used as a "toy" model to study the confinement problem [25]. The bound state spectrum of electrodynamics in (2+1) dimensions has been studied, and the Bethe-Salpeter equation for the bound states has been solved using the quenched ladder approximation and shown to display confining behaviour [26]. Also, the (2+1)D theory is known to display the finite-temperature behaviour of the corresponding (3+1)D theory [27, 28]. In any case, electrodynamics in (2+1)D should be applicable to electrodynamic surface effects.

Field theory in (2+1) dimensions displays many unusual properties. They can be unique to (2+1)D and also quite at odds with our preconceptions from (3+1)D. First, in (2+1)D the statistics are arbitrary [29–31]. This is because in two space dimensions the particle configuration space is multiply connected, so when two particles are interchanged, the wave function need not change phase by integer multiples of π, as they must in (3+1) dimensions. Such particles are known as anyons [29, 30, 32, 33], and will be discussed presently. For massless particles in 2+1 dimensions, spin is also arbitrary [34, 35]. Since spatial rotations have only a single generator, J_3, the algebra $[J_3, J_3] = 0$ cannot lead to any obvious quantization. Another peculiarity that we encounter is that in odd dimensions parity is different. Since we have an even number of spatial dimensions, the normal inversion of the position vector $x \to -x$ will correspond to a rotation, so instead we must define parity as inversion of all but the last spatial coordinate [35]. It is this which leads to parity-odd objects, such as the gauge invariant Chern-Simons term, which as we will see has a profound effect on our theory.

The simplest theory to consider in (2+1) dimensions is just quantum electrodynamics, beginning with the usual $F_{\mu\nu}F^{\mu\nu}$ Lagrangian. The problem is that when we undertake perturbation calculations, we encounter infrared (IR) divergences. When experienced in (3+1)D, this "IR catastrophe" [36] is handled by also considering processes which include the emission of soft photons. The "catastrophe" becomes untenable in (2+1)D, as it introduces nonanalytic divergences, intractable within perturbation theory.

We can understand why such IR divergences arise in (2+1)D by considering a free field theory in 2+1 space-time dimensions,

$$\int d^3x\,[((\partial\phi)^2 - \mu^2\phi^2)/2 + \bar{\psi}(i\gamma.\partial - m)\psi - F^{\mu\nu}F_{\mu\nu}/4], \tag{1.1}$$

where typically ϕ is a scalar, ψ is a spinor and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is a Maxwell gauge field. The dimensionlessness of the action (in natural units) specifies the mass dimensions of the fields,

$$[\phi], [A] \sim M^{1/2}, \qquad [\psi] \sim M^1,$$

and with interaction Lagrangians like

$$\int d^3x\,[e\bar{\psi}\gamma.A\psi + (\partial_\mu\phi)^\dagger(ieA^\mu\phi)] \tag{1.2}$$

we find that for $D = 3$, the coupling constant e has dimension $[e] \sim M^{1/2}$. A renormalizable theory is one which has only a finite number of divergent Green's functions. Electrodynamics in (2+1)D is called a super-renormalizable theory since its coupling constant e has units of $\sqrt{m}$, so since the perturbation expansion is in terms of powers of e^2, higher-order diagrams become necessarily less ultraviolet (UV) divergent, resulting in only a finite number of UV divergent *diagrams.* This very feature, which minimizes the need for renormalization of UV singularities, leads to our IR problems. As Jackiw and Templeton noted [37], higher-order terms must result in terms containing higher powers of coupling constant divided by higher powers of external momentum. Subsequently, when calculating some further diagram which contains the first result as a subgraph, and attempting further momentum integrations, the inserted result with a high power of momentum in the denominator will add to the degree of IR divergence of the momentum integral, leading inevitably to IR divergences. It was as a result of this failure by perturbation theory to handle this IR "catastrophe" that researchers turned to non-perturbative techniques.

Cornwall and co-workers [38, 39] made one of the first attempts to overcome this difficulty. To begin with, they considered a version of the theory where the gamma matrices were parity-doubled 4×4 matrices. This meant that instead of using the ordinary 2×2 gamma matrices, which would have resulted in fermions whose masses violate parity, they embedded two species of fermions, with mass terms of opposite sign, into 4×4 matrices, restoring the parity invariance of the massive Lagrangian. They then used the gauge technique *ansatz* [40] to solve the Dyson-Schwinger equations [1,41–43] giving the gauge technique equation for the fermion spectral function. They evaluated the fermion self-energy perturbatively, i.e. with a bare photon propagator, found an initial approximation for the propagator, then obtained a finite solution which now broke the *chiral* symmetry of the theory. It has since been found [44] by comparing this theory with the 2×2 version (see below) [37, 45, 46], that the zero bare mass demands $\mathcal{P}$ and $\mathcal{T}$ conservation, forcing this chiral symmetry breaking solution to be discarded.

Jackiw and Templeton [37] took a different approach. They resorted to using the ordinary 2×2 gamma matrices, which are proportional to the Pauli spin matrices, and studied massless fermions to avoid generating a photon mass. They found that in order to stop the IR catastrophe from occurring, the photon propagator needed to be "softened" , that is its IR behaviour needed to go from being of $\mathcal{O}(1/k^2)$ to $\mathcal{O}(1/k)$. Instead of using the bare photon, they considered the Dyson-Schwinger equation for the photon. This equation relates the full photon to a diagram involving full vertices and propagators [1, 41–43], and is correct to any chosen order of expansion. By truncating at a suitable level and obtaining an approximation which permitted intermediate states to influence the photon's behaviour, they were able to obtain an IR finite answer. Their method was effectively allowing the photon to be "clothed" in a cloud of *massless* fermions, moving outside perturbation theory by generating terms which were non-analytic in e. Guendelman and Radulovic and others, using both perturbative [47, 48] and non-perturbative [49] techniques, also sought to avoid these IR problems by dressing the photon propagator. They also wished to avoid the occurrence of terms that were non-analytic in e^2. To this end they exploited the residual gauge degree of freedom to eliminate the leading IR poles, resulting in a loop expansion which was analytic in the coupling constant. They found a limitation in their approach, however, since the extra vector field introduced by them was not sufficient to cure all the IR divergences, and quartic and higher-order terms in that vector field would need to be introduced at higher orders.

Practitioners of the ladder or $1/N$ expansion (where N is the number of fermions) also considered this problem [27, 28, 50–54], applying their non-perturbative scheme to it. By resumming the expansion in terms of $1/N$ they found that the IR behaviour of the photon was softened and the theory rendered IR finite. The problem was that this $1/N$ technique attempts to solve the DS equations in their nonlinear form, making analytic results at even the lowest order extremely difficult to obtain. This deficiency was seen by de Roo and Stam [55, 56],who wished to find an alternative solution to the DS equations. They saw that the gauge technique exploits an *ansatz* which renders the DS equations linear, and hence more easily explored, and attempted to apply the gauge technique in (2+1)D but neglected to heed the advice of Jackiw and Templeton [37] in using the dressed propagator. Needless to say, they found that the IR catastrophe persisted, so they went on to explore an alternative akin to that of Guendelman and Radulovic [47, 48], by introducing a gauge transformation to eliminate the leading IR poles. In order to test the effectiveness of the gauge technique together with a dressed photon propagator in solving the IR catastrophe, Waites and Delbourgo [57] considered the problem in a more systematic way, and were able to obtain an IR finite solution, without the need of any extra terms involving powers of vector fields. The solution obtained contained the lowest-order perturbation theory results within it, and gave the exact IR behaviour in the scalar and spinor versions of the theory.

Several of the calculations described above could have allowed fermion masses into the theory, since it is possible to introduce such parity-conserving fermion masses when considering the form of the theory exploiting the doubled 4×4 gamma matrices, but in the work of Jackiw and Templeton [37] and Redlich [45, 46], fermion masses would dynamically introduce a parity-violating photon mass term into the theory, which was, at the time, considered disadvantageous. This parity-violating theory [58–61] has subsequently become the focus of a huge amount of interest. The theory exploits the fact that we can introduce directly into the Lagrangian another gauge-invariant term of the form

$$\epsilon_{\mu\nu\lambda} F^{\mu\nu} A^{\lambda},$$

namely the Chern-Simons (CS) Lagrangian [62, 63], which makes it topologically non-trivial. Several later works have gone on to consider the pure CS theory, that is, CS theory with no Maxwell term present. This theory is found [64, 65] to be exactly soluble and to permit an understanding of the Jones polynomial [66,67] of knot theory in (2+1)D. The observables of this theory are Wilson lines, and the vacuum expectation values of these Wilson lines can be used to define link polynomials [64, 68–71]. Further, these results have been used to explore conformal field theory (CFT) in 2D. For a CS theory defined on a compact 2D space, the states in the Hilbert space correspond to the conformal blocks of the appropriate 2D rational conformal field theory [64, 72–74]. Another correspondence has been found, namely that the CS gauge theory is equivalent to the current algebra of the CFT [64,75,76]. This connection can then be used to classify 2D CFTs, since any CFT can be obtained by selecting the appropriate gauge group of the CS theory, and it has been conjectured that all conformal theories can be classified in this way [76].

One of the interesting features of field theory in (2+1)D is that it allows for the existence of particles with generalized statistics, known as anyons [29, 30, 32, 33]. The possibility that such particles may actually exist led researchers to consider their possible applications. It was found that anyons were precisely what was needed to explain the excitations with fractional statistics observed in the fractional quantum Hall effect [77–79]. It has also been suggested [80–82] that anyons possess some of the attributes of high T_c superconductors. The quantum mechanics of anyon systems is precisely described in terms of CS gauge theory [83, 84].

The study of gauge theories such as CS theory often lead us to the calculation of momentum integrals, and one is then confronted with UV divergences. These divergences are overcome by the use of a regularization scheme, which identifies singularities in an explicit form. There are several schemes which have been applied to CS theory, namely Pauli-Villars regularization [60], analytic regularization [85, 86], nonlocal regularization [87] and dimensional regularization [88–92]. This thesis will in part consider a new formulation of abelian CS theory which permits a consistent application of dimensional regularization [93].

1.2 The Gauge Technique

In this section we will outline the gauge technique (GT), the non-perturbative technique we will adopt to help overcome the IR problems encountered in (2+1) dimensional field theory.

The GT was originally introduced by Salam and Delbourgo [94, 95] in the early sixties. They set up an iterative technique which made consistent use of the Ward-Green-Takahashi (WGT) identities [96–98], ensuring that the scheme preserved gauge covariance at any order, then solved the Dyson-Schwinger equations [1, 41–43] for the source propagator, taking two-particle unitarity as their starting point. They found that the GT improved the UV behaviour of Feynman integrals, removing the need for a $\lambda\phi^{\dagger 2}\phi^2$ counterterm in scalar electrodynamics (SED), and they also managed to render vector electrodynamics (VED) renormalizable, which is impossible within perturbation theory. Strathdee went on [99] to use the GT to explore non-perturbative behaviour in spinor electrodynamics (QED). The problem with the GT at this stage was that since the DS equations remained in a non-linear form, it became difficult to obtain analytic solutions at higher orders, so the technique remained largely unexploited. It was not until 1977 that Delbourgo and West [40] reformulated the GT, using the Lehmann spectral representation [100–102] for the fermion and the WGT identities to obtain a simple *ansatz* for the 3-point photon-amputated Green's function which amazingly rendered the DS equations linear, resulting in the first-order GT equation for the fermion spectral function in covariant-gauge electrodynamics. They obtained a solution of this equation in the Landau gauge, and Slim [103] subsequently obtained a solution for an arbitrary covariant gauge. These successes, and the fact that the GT yielded almost trivially the *exact* IR behaviour in QED, SED and VED in covariant gauges [104, 105], prompted extensive research into applications of the GT.

The gauge properties of the GT solutions in (3+1)D were studied by Delbourgo and Keck [106], Slim [103] and Delbourgo, Keck and Parker [107], using the Zumino identity for two-point Green's functions to obtain a relationship between the spectral function in different gauges. It was found [106] that in SED, the solution obtained using this gauge covariance relation for the spectral function, and that obtained by the GT in an arbitrary gauge agreed precisely. In QED however, the spectral functions obtained from the GT only satisfied the covariance relation in the asymptotic limits [103,107], violating the Zumino identity at intermediate momenta (in sharp contrast with perturbation theory), thought to be due to the neglect of transverse amplitudes.

The GT has also been applied to lower dimensional models. Delbourgo and Shepherd applied the naive *ansatz* to the Schwinger model in the Feynman gauge [108], and returned the conventional result, with the gauge symmetry being dynamically broken. This model was considered for arbitrary gauge by

Gardner [109], who found that the naive *ansatz* was no longer consistent, and so introduced a transverse component to solve the problem. Delbourgo and Thompson [110] then showed that this transverse part of the *ansatz* was unique and complete in (1+1)D. They went on to study the Thirring model, which showed that it is possible to apply the GT to a non-gauge theory, as long as it possesses gauge-type identities. Thompson also applied the GT to a (1+1)D axial model [111], where a complete solution was possible. The GT has also been used to address the question of dynamical symmetry breaking in various models [112–114]. The results have agreed with those obtained by other methods [115], with the benefit that the GT managed to avoid the divergences found in these methods.

Given these successes in various abelian theories it is natural to want to use the GT in QCD, where non-perturbative effects are known to be important. The difficulty is that the GT utilizes the simplicity of the abelian WGT identities and the Lehmann spectral representation to obtain a very simple *ansatz*. In the non-abelian theory, the generalization of the WGT identities, the Slavnov-Taylor (ST) identities [116, 117] are more complicated, as they are influenced by the presence of ghosts. Their form, which is no longer a simple difference of propagators, is not suitable for constructing the GT *ansatz*. This difficulty has been overcome most successfully [118–121] by considering that any physical process, such as quark scattering via a single gluon exchange, must be gauge-invariant. This implies that if we were to consider all contributions to the gluon self-energy, including those which appear to be of higher-order such as multiple-gluon emissions from a single point, the "self-energy" resulting from this resummation would be gauge-invariant. Obtaining resummed propagators and vertices in this way, it can be shown [118, 121] that since the gauge dependence has become trivial (only persisting in the bare gluon propagator), the ST identities become abelian-like, which allows the GT *ansatz* to be constructed. This technique, the so-called pinch technique, has been used to show interesting features within QCD, such as dynamical gluon mass generation [118] and the prediction of the β function for the running charge, which is not summable perturbatively [121].

Despite all these successes of the lowest-order GT, there remained a limitation. When considered to only this order, it did not allow for the determination of the transverse components of vertices. This limitation had been noted and expounded upon by many researchers. In the IR region it is no limitation, since transverse effects disappear in electrodynamics at least, but in general these contributions need to be considered. In (3+1)D spinor electrodynamics, the renormalizability of the GT equation was not apparent, and it had been conjectured [122,123] that transverse corrections would remove the divergences. It was also thought that the non-gauge-covariance of the spectral function in spinor electrodynamics was due to the absence of these transverse components. This led to the consideration of an extension to the GT, which began when King [124] modified the *ansatz* in the spinor theory, introducing a transverse part. Beginning with perturbation theory, and being correct asymptotically up to leading logs, the transverse vertex refined the GT. Standard results were obtained in the asymptotic region, but the refined GT was still unable to reproduce $\mathcal{O}(e^4)$ perturbation theory. In search of a more satisfactory way of improving the GT, Parker [125, 126] considered a new approach. Looking at the scalar theory, the DS equation for the three point function was used as the starting point and a non-perturbative transverse vertex constructed which was consistent with perturbation theory and correct in any momentum region. The only limitation with this technique was that it was valid only for the Feynman gauge, and that it incorporated an arbitrary constant. Delbourgo and Zhang [127, 128] completed the refinement of the GT. They managed to generalize the work of Parker to be valid in arbitrary gauge, and also to encompass the spinor theory. Their new GT equations were finite, linear in the spectral function, exact to $\mathcal{O}(e^4)$ in any gauge, had no ambiguous constant, and gave the correct IR solution.

1.3 Structure of the Thesis

This thesis consists of six chapters, the first of which is an introduction to field theory in 2+1 dimensions and the gauge technique.

The main body of the thesis begins in Chapter 2, where a scalar version of electrodynamics in (2+1)D is considered. A framework is established which permits both perturbative and non-perturbative study of the theory. The perturbation approach is seen to be deficient in handling the infrared problems inherent in such theories, so the gauge technique is used as a non-perturbative tool to study the theory. It is found that only by dressing the photon propagator [37] can an infrared finite result be obtained. In order to understand the gauge properties of the resulting meson spectral function, the gauge covariance relation (which links the function in different gauges) is obtained, which confirms that the meson spectral function is indeed gauge-invariant.

In Chapter 3, the full spinorial version of QED is considered, and the calculations of Chapter 2 are repeated in this theory, with similar findings. We are once again required to adopt a non-perturbative approach and dress the photon propagator in order to obtain an infrared finite result. The gauge behaviour of the resulting fermion spectral function is once again explained by deriving gauge covariance relations in the spinor theory.

Chapter 4 begins our study of theories which permit the notion of parity violation. In the presence of a Chern-Simons term in the Lagrangian we see that even techniques such as dimensional regularization, which seem universally applicable, have difficulty being applied. We forego the usual naive "solution" to this problem, which involves an unnatural splitting of the D-dimensional space, and instead develop a reformulation of the theory which exists in $2l+1$ dimensions and is consistent for arbitrary l, so that dimensional regularization may safely be applied. The perturbation expansion is considered and we find that in contrast to the previous two theories, Chern-Simons theory is infrared stable, enabling the calculation of the spectral function perturbatively.

Dynamical mass generation is the topic of Chapter 5. We consider in detail the effect of a mass term which violates the parity invariance of the theory. It is found that the presence of *either* a fermion *or* photon mass in the initial theory will engender the other when quantum corrections are considered. These ideas are then generalized, by considering the effects of parity-violating terms in arbitrary odd dimensions. The induced topological mass term is calculated in arbitrary odd dimensions, and interestingly, the purely topological theory in odd dimensions greater than three is found to be distinctive in that no one loop fermion mass is generated, due to the absence of a bare propagator for the photon.

Finally, Chapter 6 is made up of a summary of the thesis together with suggestions for further study.

In addition, at the end of the thesis, several appendices are included, giving the Feynman rules used, detailing some of the calculational techniques employed, and discussing Dirac γ-matrices in odd dimensions. Reference is made to these appendices where appropriate in the text of the thesis.

Chapter 2

Scalar Electrodynamics in (2+1)D

This chapter will begin our study of gauge theories in 2+1 dimensions by considering the electrodynamics of a scalar field. This theory has the advantage that it remains relatively simple, by avoiding the multiplication of terms encountered when taking the trace of products of γ matrices, as occurs in the spinorial version of the equivalent theory. We begin by detailing the formalism of Delbourgo [129], which considered the equivalent theory in (3+1)D, and make modifications where necessary to apply the formalism to (2+1)D. We use this framework to study the theory using perturbation theory, and see explicitly the infrared singularities encountered in such an expansion. Then we exploit the GT, which due to its non-perturbative nature is able to overcome these infrared difficulties. Finally we study the gauge covariance relations of the spectral function, to try and understand its gauge (in)dependence.

2.1 Background/Introduction

We consider a simple scalar model with the usual Maxwell Lagrangian, with no Chern-Simons term, that is, the (2+1)D counterpart of ordinary (3+1)D scalar electrodynamics (SED). The Lagrangian in this case will be of the form

$$\begin{aligned}
\mathcal{L} &= \left([(\partial_\mu + ieA_\mu)\phi]^\dagger[(\partial^\mu + ieA^\mu)\phi] - m^2\phi^\dagger\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\
&\equiv \mathcal{L}_0 + \mathcal{L}_{GF}, \qquad (2.1)
\end{aligned}$$

where ϕ is the scalar field, A^μ is the gauge field and $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength. The last term in (2.1), the gauge-fixing term $\mathcal{L}_{GF}$, is introduced to eliminate the residual gauge degrees of freedom of the action, and so permit the inversion of the gauge field propagator. From (2.1) we can generate the Feynman rules of the theory by taking functional derivatives of $\mathcal{L}$ with respect to the fields. For example, the (inverse) gauge propagator is

$$\begin{aligned}
D^{-1}_{\mu\nu} &= \frac{\delta^2\mathcal{L}}{\delta A^\mu \delta A^\nu} \\
&= -\eta_{\mu\nu}k^2 + (1-\xi)k_\mu k_\nu, \qquad (2.2)
\end{aligned}$$

which may now be safely inverted. This is done using the condition that the product of the propagator and its inverse should result in $\eta_{\mu\nu}$. Selecting a propagator consisting of all possible two-index tensor forms, each carrying an unknown constant, then solving for these constants, we obtain

$$D_{\mu\nu} = \frac{-\eta_{\mu\nu}}{k^2} + (1-\xi)\frac{k_\mu k_\nu}{k^4}. \qquad (2.3)$$

Similarly, the meson propagator and the meson-meson-photon vertex can be determined. The complete set of Feynman rules for SED is given in Appendix A.

Since this is a gauge theory, we must ensure that we preserve the gauge symmetry. One way to do this is via the Ward-Green-Takahashi (WGT) identities [96–98] connecting successive source Green functions. The WGT identities can be derived by considering the effect of a set of transformations on the generating functional, $W[J]$. We can see that $\mathcal{L}_0$ in (2.1) is invariant under these transformations, which take the form of the infinitesimal gauge variation,

$$
\begin{aligned}
A_\mu(x) &\rightarrow A_\mu(x) - \partial_\mu \Lambda(x)/e \\
\phi(x) &\rightarrow \phi(x) + i\Lambda(x)\phi(x) \\
\phi^\dagger(x) &\rightarrow \phi^\dagger(x) - i\Lambda(x)\phi^\dagger(x),
\end{aligned}
$$

where $\Lambda(x)$ is a real infinitesimal scalar function. We consider the effect these transformations have on the generating functional W, which must also be invariant under them. If we define the action S as

$$S = \int d^D x(\mathcal{L}_0 + \mathcal{L}_{GF} - \mathcal{L}_S), \tag{2.4}$$

where the source term $\mathcal{L}_S$ is given by

$$\mathcal{L}_S = j^\mu A_\mu + \eta^\dagger \phi + \phi^\dagger \eta,$$

(where j^μ, $\eta^\dagger$, η are the sources of A_μ, ϕ, $\phi^\dagger$ respectively) then the vacuum generating functional is

$$Z[j_\mu, \eta, \eta^\dagger] = \int [d\phi d\phi^\dagger dA_\mu] \exp[iS], \tag{2.5}$$

and further W, the generating functional of the Green's functions, is defined by

$$Z[j_\mu, \eta, \eta^\dagger] = \exp[iW[j_\mu, \eta, \eta^\dagger]].$$

Considering the variation of the gauge-fixing and source terms (since $\Delta\mathcal{L}_0 = 0$),

$$\Delta(\mathcal{L}_{GF} - \mathcal{L}_S) = (\partial \cdot A)(\partial^2 \Lambda)/(e\xi) - j^\mu \partial_\mu \Lambda/e - i\eta^\dagger \Lambda \phi + i\phi^\dagger \Lambda \eta,$$

and demanding the invariance of Z under this variation then implies

$$\left[\frac{\partial^2 \partial_\mu}{\xi} i \frac{\delta}{\delta j_\mu} - \partial^\mu j_\mu + e\eta^\dagger \frac{\delta}{\delta \eta^\dagger} - e\eta \frac{\delta}{\delta \eta}\right] Z = 0, \tag{2.6}$$

which is the fundamental functional gauge identity. In terms of W it takes the form

$$\left[\frac{\partial^2 \partial_\mu}{\xi} i \frac{\delta W}{\delta j_\mu} - \partial^\mu j_\mu + e\eta^\dagger \frac{\delta W}{\delta \eta^\dagger} - e\eta \frac{\delta W}{\delta \eta}\right] = 0. \tag{2.7}$$

We take the Legendre transform of (2.7) via

$$\Gamma[\phi, A_\mu, \phi^\dagger] = W[\eta^\dagger, j_\mu, \eta] + \int (A_\mu j^\mu + \eta^\dagger\phi + \phi^\dagger\eta),$$

which relates the one-particle-irreducible generating functional Γ to W, resulting in

$$\left[\frac{\partial^2\partial_\mu}{\xi} i A^\mu - \partial^\mu \frac{\delta\Gamma(x)}{\delta A_\mu} + e\frac{\delta\Gamma(x)}{\delta\phi(x)}\phi(x) - e\phi^\dagger(x)\frac{\delta\Gamma(x)}{\delta\phi^\dagger(x)}\right] = 0. \tag{2.8}$$

This contains all the information we need to obtain any of the WGT identities within this theory. To obtain the WGT identity which involves the meson propagator, we need to take the functional derivative of (2.8) with respect to $\phi(x)$ and its conjugate, i.e $\frac{\delta^2}{\delta\phi(y)\delta\phi^\dagger(z)}$, which yields

$$\partial_\mu \frac{\delta^3\Gamma}{\delta A_\mu(x)\delta\phi(y)\delta\phi^\dagger(z)} = e\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi^\dagger(z)}\delta^4(x-y) - e\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi^\dagger(x)}\delta^4(z-x).$$

Now we need to make the identification that $\frac{\delta^3\Gamma}{\delta A_\mu(x)\delta\phi(y)\delta\phi^\dagger(z)} \equiv \Gamma_\mu(x;y,z)$ is the full photon-meson-meson vertex and $\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi^\dagger(y)} \equiv \Delta^{-1}(x,y)$ is the inverse meson propagator, so our relation becomes

$$\partial_\mu\Gamma^\mu(x;y,z) = e\delta^4(z-x)\Delta^{-1}(y,x) - e\delta^4(x-y)\Delta^{-1}(x,z).$$

Finally, transforming this to momentum space, we obtain the familiar expression

$$k^\mu\Gamma_\mu(p, p-k) = \Delta^{-1}(p) - \Delta^{-1}(p-k). \tag{2.9}$$

Similarly, by taking suitable functional derivatives of (2.8) above, we may derive WGT identities for the photon field and higher-order Green's functions.

By choosing a function which satisfies its associated WGT identity, we preserve the gauge symmetry of the theory. It is possible to begin with the lowest order WGT identity in its usual form, (2.9), then solve for Γ_μ in terms of Δ^{-1}. This is the technique used in the original references on the GT [94,95,99,122] and by Ball and Chiu [130,131] and subsequent workers [132–134]. The problem with this approach is that it produces a nonlinear equation for Δ^{-1}. When substituted into the relevant Dyson-Schwinger equation, things become *very* complicated to solve, and we find ourselves no better off computationally than practitioners of the ladder approximation, which is a severe limitation. Instead we follow Delbourgo [129] and manipulate equation (2.9) by multiplying it on the left by $\Delta(p)$ and on the right by $\Delta(p-k)$, giving us

$$k^\mu \Delta(p)\Gamma_\mu(p, p-k)\Delta(p-k) = \Delta(p-k) - \Delta(p). \tag{2.10}$$

In order to set up an iterative way of solving for the particle propagators, we will utilize the Lehmann spectral representation of the meson [100–102], namely

$$\Delta(p) = \int_{-\infty}^{\infty} \frac{\varrho(\omega)d\omega}{p^2 - \omega^2 + i\varepsilon}. \tag{2.11}$$

We use this form of dispersion relation rather than the conventional

$$\Delta(p) = \int \frac{\rho(\omega^2)d\omega^2}{p^2 - \omega^2 + i\varepsilon}$$

as the spectral function in three dimensions naturally takes a form involving $\sqrt{p^2}$, as will become apparent. By observing that the difference

$$\Delta(p-k) - \Delta(p) = \int \frac{(2p-k)_\mu k^\mu \varrho(\omega)d\omega}{(p^2-\omega^2)[(p-k)^2-\omega^2]}, \tag{2.12}$$

Delbourgo [129] saw that a very simple, though not unique, solution of (2.10) is to take the longitudinal Green's function as

$$\Delta(p)\Gamma_\mu(p, p-k)\Delta(p-k) = \int \frac{(2p-k)_\mu \varrho(\omega)d\omega}{(p^2-\omega^2)[(p-k)^2-\omega^2]}. \tag{2.13}$$

It is clear that this is exact only up to an arbitrary transverse function, which could be added to (2.13) without violating the gauge identities, since any transverse function will be annihilated when contracted with k^μ.

Now, to find the lowest-order corrections to the bare propagators, we consider the Dyson-Schwinger (DS) equations [1, 41–43] for the propagators. We find it convenient to work in momentum space, and if we assume $A_\mu(k)$ and $j_\mu(k)$ are the Fourier transforms of $A_\mu(x)$ and $j_\mu(x)$, and similarly for all other quantities, then we can obtain the Fourier transform of the action (2.4) explicitly, giving

$$
\begin{aligned}
S \;=\; \int \bar{d}^3k \Bigg[& -\frac{1}{4}F_{\mu\nu}(k)F^{\mu\nu}(-k) - \frac{k_\mu k_\nu A^\mu(k)A^\nu(-k)}{2\xi} + (k^2-m^2)\phi^\dagger(-k)\phi(k) \\
& -e\int \bar{d}^3p\,(p-k)_\mu \phi^\dagger(p)A^\mu(-p-k)\phi(k) \\
& +e^2\int \bar{d}^3p\,\bar{d}^3p'\phi^\dagger(p)A_\mu(-p-p'-k)A^\mu(p')\phi(k) \\
& -\eta^\dagger(k)\phi(-k) - \phi^\dagger(k)\eta(-k) - j^\mu(k)A_\mu(-k)\Bigg],
\end{aligned} \tag{2.14}
$$

where we now adopt the convention that $\bar{d}^3p \equiv d^3p/(2\pi)^3$, which we will use throughout this thesis. The DS equations result from the fact that the vacuum expectation value of the functional derivative of the action with respect to any of its field operators is identically zero, for example,

$$
\begin{aligned}
0 \;&=\; \int [d\phi d\phi^\dagger dA_\mu]\left(\frac{\delta}{\delta\phi(k)}\exp[iS]\right) \\
&=\; \int [d\phi d\phi^\dagger dA_\mu]\Bigg[(k^2-m^2)\phi^\dagger(-k) - e\int \bar{d}^3p(p-k)_\mu\phi^\dagger(p)A^\mu(-p-k) \\
&\quad +e^2\int \bar{d}^3p\bar{d}^3p'\phi^\dagger(p)A_\mu(-p-p'-k)A^\mu(p') - \eta^\dagger(-k)\Bigg]\exp[iS]\;.
\end{aligned} \tag{2.15}
$$

Noting from (2.5) that

$$
i\frac{\delta Z}{\delta\eta(p)} = \int [d\phi d\phi^\dagger dA_\mu]\phi^\dagger(-p)\exp[iS]
$$

and

$$
i\frac{\delta Z}{\delta j_\mu(p+k)} = \int [d\phi d\phi^\dagger dA_\mu]A_\mu(-p-k)\exp[iS],
$$

we can express (2.15) as

$$
\begin{aligned}
\Bigg(& i(k^2-m_0^2)\frac{\delta}{\delta\eta(k)} - \eta^\dagger(-k) - e_0\int \bar{d}^3p(p-k)_\mu\frac{\delta^2}{\delta\eta(-p)\delta j^\mu(p+k)} \\
& +ie_0^2\bar{d}^3p\bar{d}^3p'\frac{\delta^3}{\delta\eta(-p)\delta j^\mu(p+p'+k)\delta j_\mu(-p')}\Bigg) Z[\eta,\eta^\dagger,j_\mu] = 0,
\end{aligned}
$$

which in terms of the generating functional W is

$$
\begin{aligned}
&(m_0^2-k^2)\frac{\delta W}{\delta\eta(k)}-\eta^\dagger(-k)\\
&+e_0\int d\!\!^{-3}p\,(p-k)_\mu\left[i\frac{\delta^2 W}{\delta\eta(-p)\delta j^\mu(p+k)}-\frac{\delta W}{\delta\eta(-p)}\frac{\delta W}{\delta j^\mu(p+k)}\right]\\
&+ie_0^2\int d\!\!^{-3}p\,d\!\!^{-3}p'\left[\frac{\delta W}{\delta\eta(-p)}\frac{\delta^2 W}{\delta j^\mu(p+p'+k)\delta j_\mu(-p')}+\frac{\delta^3 W}{\delta\eta(-p)\delta j^\mu(p+p'+k)\delta j_\mu(-p')}\right.\\
&-i\left(\frac{\delta^2 W}{\delta\eta(-p)\delta j^\mu(p+p'+k)}\frac{\delta W}{\delta j_\mu(-p')}+\frac{\delta W}{\delta j^\mu(p+p'+k)}\frac{\delta^2 W}{\delta\eta(-p)\delta j_\mu(-p')}\right.\\
&\left.\left.+\frac{\delta W}{\delta\eta(-p)}\frac{\delta^2 W}{\delta j^\mu(p+p'+k)\delta j_\mu(-p')}\right)\right]=0. \qquad (2.16)
\end{aligned}
$$

Equation (2.16) may be used to generate the DS equation for any photon-amputated Green's function G. For example, if we wish to generate the DS equation for the meson, we take the functional derivative of (2.16) with respect to $\eta^\dagger(q)$. We then set the sources to zero, and note that

$$
\frac{W(0)}{\delta j^\mu}=\frac{W(0)}{\delta\eta}=\frac{W(0)}{\delta\eta^\dagger}=0,
$$

due to the absence of spontaneous breaking of charge symmetry or Lorentz invariance. This results in

$$
\begin{aligned}
(m_0^2-k^2)\frac{\delta^2 W}{\delta\eta^\dagger(q)\delta\eta(k)}-1 &= +ie_0\int d\!\!^{-3}p\,(p-k)_\mu\frac{\delta^3 W[0]}{\delta\eta^\dagger(q)\delta\eta(-p)\delta j^\mu(p+k)}-\\
&-e_0^2\int d\!\!^{-3}p\,d\!\!^{-3}p'\left[i\frac{\delta^2 W[0]}{\delta\eta^\dagger(q)\delta\eta(-p)}\frac{\delta^2 W[0]}{\delta j^\mu(p+p'+k)\delta j_\mu(-p')}+\right.\\
&\left.+\frac{\delta^4 W[0]}{\delta\eta^\dagger(q)\delta\eta(-p)\delta j^\mu(p+p'+k)\delta j_\mu(-p')}\right]. \qquad (2.17)
\end{aligned}
$$

We now define the $(n+2)$-point unrenormalized Green's function (with n external photon lines) by

$$
(W_u)^{\mu_1\cdots\mu_n}(p',\ldots;p,\ldots)\bar\delta(p+\ldots-p')=i^{n+3}\frac{\delta^{n+2}W[0]}{\delta\eta(-p')\delta\eta^\dagger(p)\delta j_{\mu_1}\ldots\delta j_{\mu_n}},
$$

in terms of which (2.17) becomes (after integrating out the δ functions)

$$
\begin{aligned}
1 &= (k^2-m_0^2)\Delta_u(k)+ie_0\int d\!\!^{-3}p\,(p-k)_\nu(W_u)^\nu(k,p)+ie_0^2\Delta_u(k)\int d\!\!^{-3}p\,g_{\mu\nu}(D_u)^{\mu\nu}(p)\\
&-ie_0^2\int d\!\!^{-3}p\,d\!\!^{-3}p' g_{\mu\nu}(W_u)^{\mu\nu}(p,p',k-p-p';k), \qquad (2.18)
\end{aligned}
$$

where the u subscripts denote unrenormalized quantities. We wish to write this in terms of the photon-amputated Green's functions, $G \equiv \Delta\Gamma\Delta$, which are defined by

$$(W_u)^{\mu_1\cdots\mu_n}(p',\ldots;p,\ldots)\bar{\delta}(p+\ldots-p') = \\ i^{n+3}(-e_0)^n(D_u)^{\mu_1\nu_1}\ldots(D_u)^{\mu_n\nu_n}(G_u)_{\nu_1\ldots\nu_n}(p'\ldots p\ldots),$$

and using this we obtain

$$\begin{aligned} 1 &= (k^2-m_0^2)\Delta_u(k) - ie_0^2\int \bar{d}^3p\ (p-k)_\nu(D_u)^{\mu\nu}(k-p)(G_u)\nu(k,p) \\ &+ie_0^2\Delta_u(k)\int \bar{d}^3p\ g_{\mu\nu}(D_u)^{\mu\nu}(p) \\ &+e_0^4\int \bar{d}^3p\ \bar{d}^3p'(D_u)^{\nu\alpha}(p')(D_u)^{\nu\beta}(k-p-p')(G_u)_{\alpha\beta}(p,p',k-p-p';k). \end{aligned} \tag{2.19}$$

If we renormalize this equation multiplicatively, using

$$\Delta^{-1} = Z_\phi\Delta_u^{-1}, \quad \Gamma^\mu = Z_\phi\Gamma_u^\mu, \quad D^{\mu\nu} = Z_A^{-1}D_u^{\mu\nu}, \quad e = Z_A^{1/2}e_0,$$

and write it in terms of the vertex functions (or Γ's), we achieve the meson DS equation given in (2.21) below. A similar approach would also yield the photon DS equation.

The DS equations are not part of perturbation theory, as they involve full propagators instead of a bare loop expansion, but they are consistent with it to any order of expansion in e, and the lowest-order perturbation result can be regained by putting $\varrho^{(0)}(\omega) = \delta(\omega - m)$. We adopt this form only to allow a consistent approach in the next two sections. The first in the infinite series of complete DS equations for the photon reads

$$\begin{aligned} D_{\mu\nu}^{-1}(k) &= (-\eta_{\mu\nu}k^2 + (1-\xi)k_\mu k_\nu)Z_A + 2ie^2Z\int \eta_{\mu\nu}\Delta(p)\bar{d}^3p \\ &\quad -ie^2Z\int \bar{d}^3p\ \Delta(p)\Gamma_\mu(p,p-k)\Delta(p-k)(2p-k)_\nu \\ &\quad +2e^4Z\int \Delta(p)\Gamma_{\kappa\nu}(p,k;p',k')\Delta(p')D_\mu^\kappa(k')\bar{d}^3p\bar{d}^3p' \\ &\equiv -(-\eta_{\mu\nu}k^2 + (1-\xi)k_\mu k_\nu)Z_A + \Pi_{\mu\nu}(k), \end{aligned} \tag{2.20}$$

where Z is the source renormalization constant, Z_A is that of the photon and $\Gamma_{\mu\nu}$ stands for the meson-photon scattering vertex with the momentum arguments stated. We may also represent equation (2.20) in terms of Feynman diagrams, which we do in Figure 1 below.

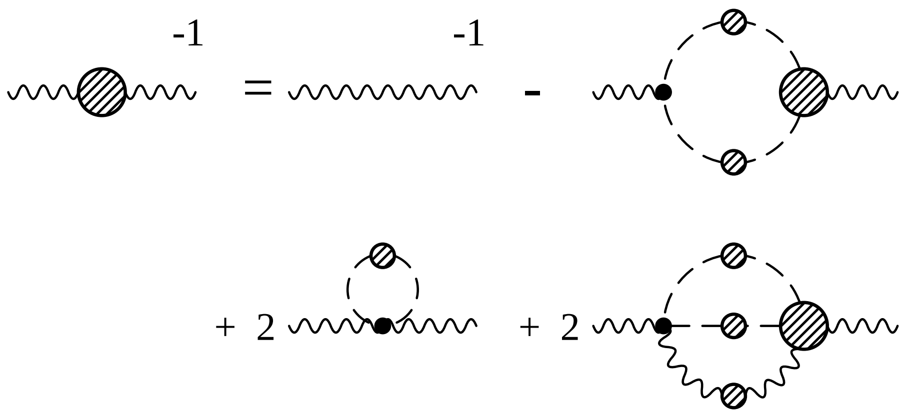


Figure 1: Photon DS equation in SED

In Figure 1, (and Figure 2 below) a wavy line corresponds to a photon, a dashed line represents a meson, a dot corresponds to a vertex, and a shaded "blob" indicates that the propagator is regarded as full, i.e. exact to all orders. Similarly, the lowest DS equation for the scalar meson (assuming for the present that it has a non-zero bare mass m_0), is

$$\begin{aligned} Z_\phi^{-1} &= (p^2 - m_0^2)\Delta(p) - ie^2 \int \bar{d}^3k \; \Delta(p)\Gamma_\nu(p, p-k)\Delta(p-k)D^{\mu\nu}(k)(2p-k)_\mu \\ &+ie^2\Delta(p)\int D_\mu^{\ \mu}(k)\bar{d}^3k \\ &+2e^4\int\Delta(p)\Gamma_{\mu\nu}(p, k; p', p+p'+k)\Delta(p+p'+k)D^{\mu\lambda}(k)D^\nu_{\ \lambda}(k')\bar{d}^3p\,\bar{d}^3p', \end{aligned} \tag{2.21}$$

which may also be represented diagramatically, as shown in Figure 2.
These equations hold to all orders, since they involve the full propagators and vertex functions, making them potentially very powerful. Within this framework, that is, using the *ansatz* (2.13) to linearize the DS equations, the photon polarization is given by

$$\Pi_{\mu\nu}(k) = -ie^2Z\int\varrho(\omega)d\omega\int\bar{d}^3p\left[\frac{(2p-k)_\mu(2p-k)_\nu}{(p^2-\omega^2)[(p-k)^2-\omega^2]} - \frac{2\eta_{\mu\nu}}{p^2-\omega^2}\right] \tag{2.22}$$

$$+ \text{ 2-meson} - \text{1-photon terms}$$

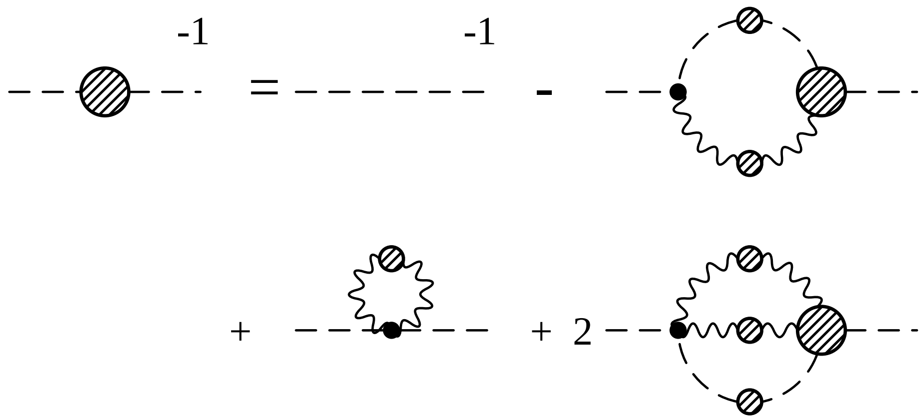


Figure 2: Meson DS equation in SED

and the meson propagator obeys the equation

$$\begin{aligned} Z_\phi^{-1} &= (p^2 - m_0^2)\Delta(p) \\ &\quad - \int \varrho(\omega) d\omega \; ie^2 \int \vec{d}^3 k \left[\frac{(2p-k)_\mu (2p-k)_\nu D^{\mu\nu}(k)}{(p^2-\omega^2)[(p-k)^2-\omega^2]} - D_\mu^{\ \mu}(k) \right] \\ &\quad + \text{2-photon -- 1-meson terms} \\ &\equiv (p^2 - m_0^2)\Delta(p) + \int \frac{\varrho(\omega) d\omega}{p^2-\omega^2} \Sigma(p,\omega), \end{aligned} \tag{2.23}$$

or, upon using the renormalization condition $m^2 = m_0^2 - \Sigma(m,m)$ [122] we find

$$0 = \int \frac{\omega^2 - m^2 + \Sigma(p,\omega) - \Sigma(\omega,\omega)}{p^2 - \omega^2 + i\varepsilon} \varrho(\omega) d\omega. \tag{2.24}$$

Since this Σ is still the full meson self energy, we must be careful in taking the imaginary part of (2.24). If we are taking the discontinuity of some integral

$$\int d\omega \frac{f(\omega)}{p^2-\omega^2},$$

then we obtain two contributions,

$$\int d\omega \frac{\Im f(\omega)}{p^2-\omega^2} \qquad \text{and} \qquad \int d\omega \Re f(\omega) \delta(p^2-\omega^2).$$

Returning to (2.24), we find that

$$\begin{aligned} \Im \left[\int \frac{\Sigma(p,\omega) - \Sigma(\omega,\omega)}{p^2-\omega^2} d\omega \right] &= \int d\omega \frac{\Sigma_I(p,\omega)}{p^2-\omega^2} - \int d\omega \frac{\Sigma_I(\omega,\omega)}{p^2-\omega^2} \\ &\quad + \int d\omega [\Sigma_R(p,\omega) - \Sigma_R(\omega,\omega)] \, \delta(p^2-\omega^2), \end{aligned} \tag{2.25}$$

where $\Sigma_I \equiv \frac{1}{\pi}\Im\Sigma(p,\omega)$ is the discontinuity of the meson self energy for a mass ω meson, and $\Sigma_R \equiv \frac{1}{\pi}\Re\Sigma$ is its real part. The second integral on the right hand side is obviously zero, since we know that the self-mass $\Sigma(m,m)$ is real, and the final term will not contribute since the δ function will cause its two parts to cancel. This means we may write the imaginary part of (2.24) as

$$\frac{(p^2-m^2)}{2\sqrt{p^2}}\varrho(p) = \int \frac{\varrho(\omega)d\omega}{p^2-\omega^2}\Sigma_I(p,\omega). \tag{2.26}$$

We now have the necessary tools to permit the consistent study of SED using both perturbative and non-perturbative techniques. We will begin in the next section, by considering perturbation theory.

2.2 Perturbation Theory

In this section we will consider the suitability of using perturbation theory to explore the properties of super-renormalizable theories, in particular (2+1)D SED.

Perturbation theory involves approximating physical quantities through a power expansion in orders of coupling constant, and summing the Feynman diagrams at each order. In (3+1)D electrodynamics it has been incredibly successful, since the small effective coupling constant results in finite terms in the perturbation expansion. We begin our study of SED in (2+1)D by considering the first order correction to the bare photon propagator, the vacuum polarization $\Pi_{\mu\nu}$, which involves the calculation of the Feynman diagram shown in Figure 3.

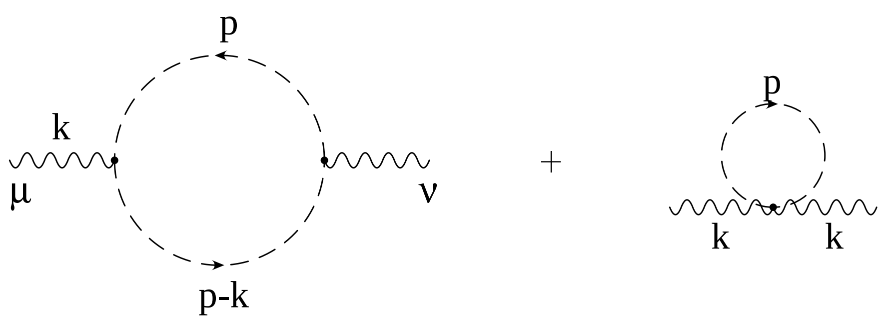


Figure 3: Photon vacuum polarization contributions in SED

As intimated in the previous section this perturbation expression is equivalent to setting $\varrho^{(0)}(\omega) = \delta(\omega - m)$ in (2.22), giving

$$\Pi_{\mu\nu}(k) = -ie^2 \int d\!\!{}^-{}^3 p \left[\frac{(2p-k)_\mu (2p-k)_\nu - 2\eta_{\mu\nu}[(p-k)^2 - m^2]}{(p^2-m^2)[(p-k)^2-m^2]} \right]. \qquad (2.27)$$

We wish to explore the UV convergence of this integral, which we can do using power counting. This is a method of seeing the superficial degree of divergence of an integral by comparing the power of momentum in the numerator and denominator. If the total power of momentum in the numerator (allowing for the dimension of the momentum integration) is larger than that in the denominator, then at large momenta the integral will diverge, whereas a larger power of momentum in the denominator will have the converse effect, yielding a UV finite result. It can be seen that in the above integral, equation (2.27), the effective momentum is $(3+2)-4 = 1$ so the numerator dominates, and the integral appears to be UV divergent. A regularization scheme is required to evaluate the momentum integral, and render any residual singularities into an amenable form, ready for renormalization techniques. We choose dimensional regularization [135–139], for several reasons. First, it is convenient, since any infinities encountered appear simply as poles in Γ functions. Also, it is simple to use, since the propagators retain their inverse quadratic form, making the integrations relatively easy to compute. Finally, it preserves the gauge invariance of the theory, which is vital if results in a general covariant gauge are required. The technique of dimensional regularization is outlined in Appendix B, where (2.27) is evaluated explicitly, giving

$$\Pi_{\mu\nu}(k) = -\frac{e^2}{16\pi}(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}) \left[4m + \left(\sqrt{k^2} - \frac{4m^2}{\sqrt{k^2}} \right) \ln \left(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}} \right) \right]. \quad (2.28)$$

Notice that the result is strictly finite, which is ensured by gauge invariance. If we study the asymptotic behaviour of (2.28) we see that, provided $m \neq 0$, as $k \to 0$, Π tends to $e^2 k^2 / 6\pi m$, and otherwise it equals $-e^2\sqrt{-k^2}/16$. Alternatively, it is possible to evaluate (2.27) using the technique outlined in Appendix C, which involves determining the discontinuity of the momentum integral, then expressing the full integral as a dispersion relation involving its discontinuity. This method also yields (2.28). It will be useful to incorporate this photon self-energy in the full photon propagator via a dispersion relation [37]. We do this by finding an asymptotic approximation of (2.28) valid both for $k \to 0$ and $k \to \infty$ and which becomes exact for $m = 0$; explicitly

$$D^{-1}_{\mu\nu} \simeq -(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})\left(k^2 + \frac{\frac{e^2}{16}k^2}{\frac{3}{2}\pi m - \sqrt{-k^2}}\right). \tag{2.29}$$

Taking the discontinuity of the inverse of this equation yields

$$\Im D_{\mu\nu}(k) = -(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})\left(\frac{e^2}{16\sqrt{k^2}(k^2+c^2)} + \frac{3m\pi}{2c}\delta(k^2)\right) , \tag{2.30}$$

where $c = \frac{3}{2}\pi m + \frac{e^2}{16}$. Then, using

$$D_{\mu\nu}(k) = -(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})\int_0^\infty \frac{\bar\rho(\mu)d\mu}{k^2-\mu^2} \tag{2.31}$$

and noting that

$$\begin{aligned}(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})\bar\rho(\mu) &\equiv \frac{2\mu}{\pi}\Im D_{\mu\nu}(\mu) \\ &= (-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})\left[\frac{2c/\pi}{c^2+\mu^2} + 3m\left(\frac{\pi}{2c}\delta(\mu) - \frac{1}{c^2+\mu^2}\right)\right],\end{aligned}$$

we obtain (up to a Z_A scale factor) the spectral representation ($m \neq 0$) of the dressed photon propagator,

$$D_{\mu\nu}(k) = (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\left[(\frac{2c}{\pi} - 3m)\int_0^\infty \frac{d\mu}{k^2-\mu^2}\frac{1}{\mu^2+c^2} + \frac{3m\pi}{2ck^2}\right] - \xi\frac{k_\mu k_\nu}{k^4}. \tag{2.32}$$

Note the dangerous pole at $k^2 = 0$ is lurking in (2.32) when $m \neq 0$.

We now turn to the meson self-energy, $\Sigma(p, m)$, within perturbation theory. This is equivalent to evaluating the Feynman diagram in Figure 4.
We obtain the expression for $\Sigma(p, m)$ from equation (2.23) by limiting ourselves to the bare photon propagator,

$$D_{\mu\nu}(k) = -\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4}(1-\xi), \tag{2.33}$$

resulting in the expression

$$\begin{aligned}\Sigma(p,m) &= -ie^2\int d\!\!\bar{}^3k\frac{(2p-k)^\mu(2p-k)^\nu}{(p-k)^2-m^2}\left[-\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4}(1-\xi)\right] \\ &\quad -ie^2\int d\!\!\bar{}^3k\frac{1+\xi}{k^2}\,.\end{aligned} \tag{2.34}$$

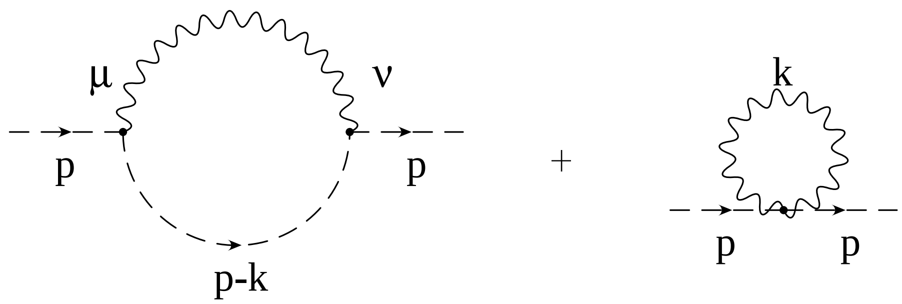


Figure 4: Contributions to meson self-energy in SED

The second integral in equation (2.34) obviously disappears in dimensional regularization, since within that scheme,

$$
\begin{aligned}
-i\int \frac{\bar{d}^3k}{k^2} &= \lim_{\substack{T\to -1\\ \Sigma\to 0}} -i\int \frac{\bar{d}^3k(k^2)^T}{(k^2-M^2)^\Sigma} \\
&= \lim_{\substack{T\to -1\\ \Sigma\to 0}} \frac{(-1)^{T-\Sigma}\Gamma(l+T)\Gamma(\Sigma-l-T)}{(4\pi)^l\Gamma(l)\Gamma(\Sigma)(M^2)^{\Sigma-l-T}} \\
&= 0. \qquad (2.35)
\end{aligned}
$$

We now evaluate (2.34) using the techniques associated with dimensional regularization, yielding

$$
\Sigma(p,m) = \frac{e^2(p^2+m^2)}{4\pi\sqrt{p^2}}\log\left[\frac{m+\sqrt{p^2}}{m-\sqrt{p^2}}\right] - \frac{e^2 m}{2\pi}, \qquad (2.36)
$$

the imaginary part of which is

$$
\Sigma_I(p,m) = \frac{e^2}{4\pi\sqrt{p^2}}(p^2+m^2)\theta(p^2-m^2), \qquad (2.37)
$$

where $\theta(p^2-m^2)$ is just the unit step function, defined by

$$
\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}
$$

Notice that both (2.36) and (2.37) happen to be *gauge independent* in three dimensions. The explanation for this is given in Appendix C, where we summarize the relevant calculations in any dimension.

Since Σ_I in (2.37) above is of order e^2, equation (2.26) may be used iteratively to give the perturbation expansion for ϱ, taking the form

$$
\varrho(\omega) = \sum_{i=0}^{\infty} \varrho^{(i)}(\omega).
$$

Putting the ith order expression for ϱ in the right hand side of (2.26) will give the $(i+1)$th order term in the expansion on the left hand side If we now do this, by beginning with the lowest-order result, $\varrho^{(0)}(\omega) = \delta(\omega - m)$, we arrive at the first order (e^2) spectral function

$$\varrho^{(1)}(p) = \frac{e^2}{2\pi} \frac{p^2 + m^2}{(p^2 - m^2)^2} \theta(p^2 - \omega^2) \ . \tag{2.38}$$

Attempting to carry out the perturbation expansion to the next order shows how things can go wrong if we do not allow for photon-line corrections in our $\mathcal{O}(e^4)$ calculations. We are confronted with the following integral

$$(p^2 - m^2)\varrho^{(2)}(p) = \frac{e^4}{4\pi^2} \int_m^p \frac{(p^2 + \omega^2)(\omega^2 + m^2)}{(p^2 - \omega^2)(\omega^2 - m^2)^2} d\omega \tag{2.39}$$

which is clearly divergent at both ends of integration, and near $\omega = m$ we meet the so-called "infrared catastrophe". In (2+1)D it is more like a "cataclysm" since unlike SED in (3+1)D, the divergence is not logarithmic but linear.

2.3 The Gauge Technique

In this section, we will consider non-perturbative methods to try to overcome this IR difficulty. The first alternative is that it may be possible to continue studying (2.26), but instead of a perturbation expansion, recast (2.26) into the form ($m = 0$)

$$p\frac{\varrho(p)}{2} = \int \frac{\Sigma_I(p,\omega) - \Sigma_I(p,p)}{p^2 - \omega^2} \varrho(\omega) d\omega + \Sigma_I(p,p)\Delta(p), \tag{2.40}$$

then try a power law selection of $\Delta(p)$ to avoid the singularity. Equation (2.40) looks to be in a more well-behaved form, but our work suggests that this naive hope is unlikely to succeed as we find that a $p = \omega$ singularity in the integration region persists.

The only way we know to effect a cure is to exploit the GT and also, following Jackiw and Templeton [37], allow the photon propagator to become dressed in order to cure the divergences of equation (2.39). In this context we use the WGT identity in the form of (2.13) to determine the photon propagator, and go on to evaluate the meson self-energy using this dressed photon propagator. Evaluating $\Pi_{\mu\nu}(k)$ from (2.22) we firstly obtain a non-perturbative estimate of the photon self-energy,

$$\Pi^T_{\mu\nu}(k) = \frac{e^2}{16\pi}(\frac{k_\mu k_\nu}{k^2} - \eta_{\mu\nu})\int d\omega \varrho(\omega)\left[4\omega + \left(\sqrt{k^2} - \frac{4\omega^2}{\sqrt{k^2}}\right)\log\left(\frac{2\omega + \sqrt{k^2}}{2\omega - \sqrt{k^2}}\right)\right], \quad (2.41)$$

before attempting to determine the meson self-energy. Since as yet we know nothing about the non-perturbative behaviour of $\varrho(\omega)$, we will assume a finite mass m threshold as a starting point and put $\varrho(\omega) = \delta(\omega - m)$ to return to the perturbation result (2.28) for Π. Later, having determined the behaviour of $\varrho(\omega)$, we may return to (2.41) to refine our result, since the dressing of the photon line is the *only* source of nonlinearity in the GT.

Let us see why the question of mass is so important in our calculations. Assume for a moment that we take $m \neq 0$, which means using the massive dressed version (2.32) of the photon propagator. This would result in our meson self-energy taking the form

$$\begin{aligned}\Sigma(p,m) &= -ie^2 \int d\!\!\!^-{}^3k \frac{(2p-k)^\mu (2p-k)^\nu}{(p-k)^2 - m^2} \times \\ &\times \left[(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\left((\frac{2c}{\pi} - 3m)\int_0^\infty \frac{d\mu}{k^2-\mu^2}\frac{1}{\mu^2+c^2} + \frac{3m\pi}{2ck^2}\right) - \xi\frac{k_\mu k_\nu}{k^4}\right],\end{aligned}$$

which, after some calculation yields a discontinuity

$$\begin{aligned}\Sigma_I(p,\omega) &= \frac{-e^2}{4\pi\sqrt{p^2}}(3p^2+\omega^2) + \frac{e^4}{32\pi^2\sqrt{p^2}}\left[\frac{\pi(p^2-\omega^2)^2}{2c^3} - \frac{(p^2-\omega^2)^2}{c^2(\sqrt{p^2}-\omega)} + \right. \\ &\left. + \left(\frac{p^2-\omega^2}{c}\left(1 - \frac{p^2-\omega^2}{c^2}\right)\right)\arctan(\frac{p^2-\omega^2}{c})\right]. \quad (2.42)\end{aligned}$$

Notice that once again this result is gauge-invariant. Also, it is important to notice that only the first term in (2.42) lacks a factor of $p^2 - \omega^2$. This means that when we insert (2.42) into (2.26), only this part will retain a factor of $p^2 - \omega^2$ in the denominator. It was this factor that led to our IR problems in perturbation theory, so with one such term, and no others to cancel it, we see that we will still have an IR catastrophe. This is simply a reflection of the fact that if the source spectral function has support *away* from the origin, the low-energy part of Π will still be proportional to k^2 and contribute to the photon renormalization constant Z_A without softening the $k \to 0$ behaviour.

It seems that our only hope to effect a cure is to assume the existence of some massless intermediate state in Π. Let us therefore fix upon some scalar source with renormalized mass $m = 0$, which clothes the bare photon propagator to

$$D_{\mu\nu}(k) = (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\frac{2c}{\pi}\int_0^\infty \frac{d\mu}{k^2-\mu^2}\frac{1}{\mu^2+c^2} - \xi\frac{k_\mu k_\nu}{k^4}; \qquad c = e^2/16. \quad (2.43)$$

Using such a dressed photon propagator[a] and dropping the 2-photon–1-meson graphs which are separately gauge-invariant, our meson self-energy discontinuity becomes

$$\begin{aligned}\Sigma_I(p,\omega) \quad = \quad & \frac{e^2 c}{4\pi^2\sqrt{p^2}}\left[-\frac{2p^2+2\omega^2+c^2}{c}\arctan\left(\frac{\sqrt{p^2}-\omega}{c}\right)\right. \\ & +\frac{(p^2-\omega^2)^2}{c^3}\left\{\frac{\pi}{2}-\arctan\left(\frac{\sqrt{p^2}-\omega}{c}\right)\right\} \\ & \left.+\sqrt{p^2}-\omega-\frac{(p^2-\omega^2)^2}{c^2(\sqrt{p^2}-\omega)}\right]\theta(p^2-\omega^2), \qquad (2.44)\end{aligned}$$

which once again remains independent of ξ. Notice that if we allow $c \to 0$, we find

$$\Sigma_I(p,\omega) \sim \frac{e^2}{4\pi\sqrt{p^2}}(p^2+\omega^2)\theta(p^2-\omega^2),$$

which is exactly (2.37), so the perturbation theory result is still contained within (2.44). More significantly, $\Sigma_I(p,p) = 0$, and this is an infrared panacea! Returning to (2.26) we can now attempt to solve this linear equation for the spectral function, which has the form

$$\begin{aligned}\sqrt{p^2}\frac{\varrho(p)}{2} \quad = \quad & \frac{e^2 c}{4\pi^2\sqrt{p^2}}\int_0^p \varrho(\omega)d\omega\left[\frac{1}{\sqrt{p^2}+\omega}-\frac{\sqrt{p^2}+\omega}{c^2}\right. \\ & +\frac{p^2-\omega^2}{c^3}\left\{\frac{\pi}{2}-\arctan\left(\frac{\sqrt{p^2}-\omega}{c}\right)\right\} \\ & \left.-\frac{2p^2+2\omega^2+c^2}{c(p^2-\omega^2)}\arctan\left(\frac{\sqrt{p^2}-\omega}{c}\right)\right]. \qquad (2.45)\end{aligned}$$

[a] More generally we easily see that the constant $c = Ne^2/16$, where N is the total number of charged zero-mass particles that can couple to the photon.

Due to the complicated nature of this equation a complete analytic solution isn't possible, so we look at the behaviour in various asymptotic regimes. Since at IR momenta, i.e. $(\sqrt{p^2}-m) \ll e^2$, we may make the approximation

$$\arctan\left(\frac{\sqrt{p^2}-m}{c}\right) \sim \frac{\sqrt{p^2}-m}{c} - \frac{(\sqrt{p^2}-m)^3}{3c^3} + \mathcal{O}\left(\frac{\sqrt{p^2}-m}{e^2}\right)^5,$$

the self-energy becomes

$$\begin{aligned}\Sigma_I(p,\omega) \;\simeq\; \frac{e^2 c}{4\pi^2\sqrt{p^2}}\Bigg[&\frac{-2(p^2+\omega^2)}{c^2}(\sqrt{p^2}-\omega) - \frac{2(\sqrt{p^2}+\omega)}{c^2}(p^2-\omega^2)\\ &+\frac{(p^2-\omega^2)^2}{c^3}\left(\frac{\pi}{2}-\frac{\sqrt{p^2}-\omega)}{c}\right)\Bigg]\theta(p^2-\omega^2),\end{aligned}$$

which to leading order in $(p^2-\omega^2)$ is

$$\Sigma_I(p,\omega) \sim \frac{-e^2}{\pi^2 c}(p^2-\omega^2),$$

and the equation,

$$p\frac{\varrho(p)}{2} \sim \frac{-e^2}{\pi^2 c}\int_0^p \varrho(\omega)d\omega,$$

is readily solved to give a spectral function for the meson which behaves as

$$\varrho(p) \sim (p)^{-1-2e^2/\pi^2 c}. \tag{2.46}$$

Similarly, if we study the UV behaviour of (2.45) above, i.e. assuming $(\sqrt{p^2}-m) \gg e^2$, it is quite valid to make the approximation

$$\arctan\left(\frac{\sqrt{p^2}-m}{c}\right) \sim \frac{\pi}{2} - \frac{c}{\sqrt{p^2}-m} + \frac{c^3}{3(\sqrt{p^2}-m)^3} + \mathcal{O}\left(\frac{e^2}{\sqrt{p^2}-m}\right)^5,$$

so that the self-energy takes the UV form

$$\begin{aligned}\Sigma_I(p,\omega) \;\simeq\; \frac{e^2 c}{4\pi^2\sqrt{p^2}}\Bigg[&\frac{-(p^2+\omega^2)\pi}{c} + (\sqrt{p^2}-\omega) - \frac{c\pi}{2} + \frac{2(p^2+\omega^2)}{\sqrt{p^2}-\omega}\\ &-\frac{(p^2-\omega^2)^2}{(\sqrt{p^2}-\omega)^3} + \frac{c^2}{\sqrt{p^2}-\omega)}\Bigg],\end{aligned}$$

with leading large-p behaviour

$$\Sigma_I(p,\omega) \sim \frac{-e^2\sqrt{p^2}}{4\pi}.$$

We now need merely to solve the equation

$$p\frac{\varrho(p)}{2} \sim \frac{-e^2}{4\pi p}\int_0^p \varrho(\omega)d\omega,$$

which yields the result

$$\varrho(p) \sim p^{-2}\exp(e^2/2\pi p)\ . \tag{2.47}$$

We can see that in both momentum regimes the meson spectral function remains gauge-invariant.

2.4 Gauge Covariance Relations

In order to understand why the spectral function is gauge-invariant in both the GT and to order e^2 in perturbation theory, we will now study the general x-space behaviour of the spectral function following the technique of references [106,107], yielding the gauge covariance relations in (2+1)D for the spectral function. We wish to determine the behaviour of our propagators under the gauge transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \partial_\mu\Lambda(x) \\ \phi &\rightarrow \phi\exp\left(ie\Lambda(x)\right). \end{aligned} \tag{2.48}$$

We follow Zumino [140] (using his notation) and begin by considering the generating functional Z, which transforms via (2.48) as

$$Z_\Lambda[\eta,\eta^\dagger,j^\mu] = Z_0[\eta(\exp(-ie\Lambda(x))),\eta^\dagger(\exp(ie\Lambda(x))),j^\mu]\times\exp[ie\int j_\mu\partial^\mu\Lambda dx],$$

or, in differential form,

$$i\frac{\delta Z}{\delta\Lambda} = \left(\partial_\mu j^\mu + e\eta\frac{\delta}{\delta\eta} - e\eta^\dagger\frac{\delta}{\delta\eta^\dagger}\right)Z. \tag{2.49}$$

Let us consider the gauge changes corresponding to a change in the generating functional defined by

$$\delta Z = \frac{i}{2}\int\int\frac{\delta}{\delta\Lambda}(M(x-y))\frac{\delta}{\delta\Lambda}Z, \tag{2.50}$$

where $M(x)$ is an arbitrary infinitesimal function, even in its argument. Now we can think of Z as dependent on some new function $F(x)$ in such a way that the infinitesimal change $M(x) \equiv \delta F(x)$ induces the change given in (2.50). If we now set $\eta = \eta^\dagger = 0$, we may exploit (2.49) in obtaining

$$\delta Z[0,0,j^\mu] = \frac{-i}{2} \int\int \partial_\mu j^\mu (M(x-y)) \partial_\mu j^\mu Z[0,0,j^\mu], \tag{2.51}$$

or, in finite form

$$Z^{(M)}[0,0,j^\mu] = \exp\left[\frac{-i}{2} \int\int \partial_\mu j^\mu (M(x-y)) \partial_\mu j^\mu\right] Z^{(0)}[0,0,j^\mu]. \tag{2.52}$$

The photon propagator is defined as

$$D_{\mu\nu}(x) = \frac{\delta}{\delta j^\mu}\left[\frac{1}{Z[0,0,j^\mu]} \frac{1}{i} \frac{\delta}{\delta j^\nu} Z[0,0,j^\mu]\right], \tag{2.53}$$

giving in this case

$$D^{(M)}_{\mu\nu}(x) = D^{(0)}_{\mu\nu}(x) + \partial_\mu \partial_\nu M(x), \tag{2.54}$$

or if we take the Fourier transform of this equation we finally obtain

$$D^{(M)}_{\mu\nu}(k) = D^{(0)}_{\mu\nu}(k) + k_\mu k_\nu \widetilde{M}(k), \tag{2.55}$$

where $\widetilde{M}(k)$ is the Fourier transform of $M(x)$. Similarly, we may also derive the effect of a gauge transformation on the meson propagator. In terms of the generating functional Z, the meson is defined as

$$\Delta(x,y) = \frac{-i}{Z} \frac{\delta^2 Z}{\delta\eta(y)\delta\eta^\dagger(x)}, \tag{2.56}$$

and so it varies according to

$$\begin{aligned} \Delta^{(M)}(x,y;j^\mu) \; = \; & \exp\Bigg(ie^2[M(x-y) - M(0)] + \\ & + ie^2 \int [M(x-z) - M(y-z)] \partial^\mu j_\mu dz \Bigg) \Delta^{(0)}(x,y;j^\mu), \end{aligned} \tag{2.57}$$

or when we then set $j = 0$

$$\Delta^{(M)}(x) = \exp\left(ie^2[M(x) - M(0)]\right) \Delta^{(0)}(x), \tag{2.58}$$

as was found previously by various authors [140–144].

Using the Lehmann spectral representation

$$\Delta^{(M)}(x) = \int \rho^{(M)}(\omega)\Delta_c(x,\omega)d\omega, \tag{2.59}$$

we can recast (2.58) into the form

$$\begin{aligned}\Delta^{(M)}(x) &= \int \rho^{(M)}(\omega)\Delta_c(x,\omega)d\omega \\ &= \exp(ie^2[M(x)-M(0)])\int \rho^{(0)}(\omega)\Delta_c(x,\omega)d\omega. \end{aligned} \tag{2.60}$$

Formally we know $\widetilde{M}(k) = -\xi/k^4$, and using this we can readily find the gauge factor exponent of the meson propagator [106,107] in three dimensions by taking the Fourier transform

$$\begin{aligned} e^2[M(x)-M(0)] &= -e^2\int đ^3k e^{ik.x}\frac{\xi}{k^4} = \frac{-e^2\xi}{4(3/2-2)}\int đ^3k\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k_\mu}e^{ik\cdot x}\frac{\xi}{k^2} \\ &= -\frac{e^2\xi\sqrt{x^2}}{8\pi} \equiv -K\sqrt{x^2}, \end{aligned} \tag{2.61}$$

where we have recognized in (2.61) the causal massless propagator,

$$\int đ^3k\frac{e^{ik\cdot x}}{k^2} = \frac{-i\sqrt{\pi}}{4\pi^{3/2}\sqrt{-x^2}},$$

and have replaced $(\frac{\partial}{\partial k})^2$ by x^2. The divergence present in the integral of (2.61) is removed using dimensional regularization. We have also defined K, which is the constant corresponding to a choice of gauge function M. Now we are ready to take the Fourier transform of (2.60). Noting that $\Delta(p|K) = \int đ^3x e^{ip\cdot x}\Delta^{(M)}(x)$, and rewriting $\rho^{(M)}(\omega) \to \rho(\omega|K)$ to enable all gauge dependence to be expressed in terms of K, we obtain

$$\Delta(p|K) = \int d\omega\rho(\omega|0)\int đ^3x\ e^{ip\cdot x}e^{-iK\sqrt{x^2}}\Delta_c(x,\omega). \tag{2.62}$$

Since in SED, the free meson propagator (mass ω) can be written as

$$\Delta_c(x,\omega) = -\frac{e^{-\omega\sqrt{-x^2}}}{4\pi\sqrt{x^2}}, \tag{2.63}$$

and $\Delta(p)$ has a Lehmann spectral representation (in any gauge K),

$$\Delta(p|K) = \int \rho(\omega|K)\Delta_c(p,\omega)d\omega, \tag{2.64}$$

we can write (2.62) in the explicit form

$$\int \rho(\omega|K)\frac{d\omega}{p^2-\omega^2} = \int \rho(\omega|0)d\omega \int d\!\!{}^-{}^3x\ e^{ip\cdot x}e^{-iK\sqrt{x^2}}\left(-\frac{e^{-\omega\sqrt{-x^2}}}{4\pi\sqrt{x^2}}\right). \tag{2.65}$$

By making a Euclidean rotation, the x-integration is easily evaluated to be

$$-\int d\!\!{}^-{}^3x\ e^{ip\cdot x}\frac{e^{-i(K+\omega)\sqrt{x^2}}}{4\pi\sqrt{x^2}} = \frac{1}{p^2-(K+\omega)^2}, \tag{2.66}$$

so that we have

$$\int \rho(\omega|K)\frac{d\omega}{p^2-\omega^2} = \int \frac{\rho(\omega|0)d\omega}{p^2-(\omega+K)^2}. \tag{2.67}$$

From the discontinuity, we arrive at the covariance relation for the spectral function,

$$\rho(\omega+K|K) = \rho(\omega|0). \tag{2.68}$$

This covariance relation implies that $\rho(\omega|K)$ is a function only of $(\omega-K)$. For some function σ we thereby define the pole and cut contributions for any K,

$$\rho(a|b) \equiv \delta(a-b-m)+\sigma(a-b-m), \tag{2.69}$$

and end up with the invariant combination

$$\rho(\omega+K|K) = \delta(\omega-m)+\sigma(\omega-m)\,. \tag{2.70}$$

This establishes that the spectral function is independent of the gauge parameter, in agreement with perturbation theory (2.38), and the GT solutions (2.46) and (2.47). As we will soon see, the covariance relation is more involved when we come to the fermion spectral function.

Chapter 3

Spinor Electrodynamics in (2+1)D

In this chapter, we will be considering a more useful theory, namely spinorial quantum electrodynamics (QED). The present work most closely resembles that of Delbourgo and West [40], who considered the equivalent theory in (3+1)D, and once again we have made the necessary modifications to (2+1)D. We will follow the analysis of the previous chapter, and begin by setting up a framework for consistent study using perturbative and non-perturbative techniques. Then we will consider a perturbation expansion, and see its inability to generate finite IR behaviour. The next section will show how the GT together with a dressed photon propagator can solve these shortcomings, and finally we will study the gauge covariance relations of the spinor theory.

3.1 Background

In QED, the Lagrangian is of the form

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{3.1}$$

where ψ and $\bar{\psi}$ are the spinor field and its conjugate, A^μ is the gauge field and $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength. As in the scalar case, we have included the gauge-fixing term, scaled by the gauge parameter ξ, which allows us to safely invert the gauge propagator. Once again the Feynman rules are obtained by taking functional derivatives of the Lagrangian with respect to the relevant fields. These rules are given in Appendix A. There is some choice in selecting the gamma matrix structure associated with the γ^μ above. We could assume that we have N species of fermions, and use the 2×2 form of the matrices, related to the Pauli spin matrices by

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2,$$

as Jackiw and Templeton [37] and others have done. The problem here is that if we allow the fermions to acquire a mass, parity would be violated, which would then induce parity-violating photon masses which we wish (for the moment) to avoid. We can simply see the effect of the fermion mass term (which would be of the form $m\bar{\psi}\psi$) on the parity symmetry by considering a parity operation on

$$I = \int d^3x \; m\bar{\psi}(x_0, x_1, x_2)\psi(x_0, x_1, x_2). \tag{3.2}$$

In even D, the parity operator, $\mathcal{P}$, corresponds to an inversion of all the *spatial* coordinates, since that is an improper transformation. However when D is odd it should be regarded as a reflection of all the space coordinates *except the very last one*, x_{D-1}, in order to ensure that the determinant of the transformation remains negative. In 2+1 dimensions this corresponds to the unitary change,

$$\mathcal{P}\psi(x_0, x_1, x_2)\mathcal{P}^{-1} = \gamma_1\psi(x_0, -x_1, x_2). \tag{3.3}$$

Applying this transformation for spinor fields to (3.2) we find

$$\mathcal{P}I\mathcal{P}^{-1} = \int d^3x \; m\bar{\psi}(x_0, -x_1, x_2)\gamma_1\gamma_1\psi(x_0, -x_1, x_2).$$

From our definition of the Dirac γ-matrices, we find $(\gamma_1)^2 = -1$, and since we are integrating over a measure and $\bar{\psi}\psi$ is symmetric we can change $x_1 \to -x_1$, so parity is indeed violated, since $\mathcal{P}I\mathcal{P}^{-1} = -I$.

Instead of allowing this to happen, in the following we shall assume that all fermion species are doubled appropriately (with opposite sign mass terms) in such a way that the parity invariance of the massive Lagrangian can be restored [27,28]. The net effect is to enlarge the gamma matrices from 2×2 to 4×4, so that they take the form

$$\gamma^0=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix},\quad \gamma^1=\begin{pmatrix}i\sigma_1 & 0\\ 0 & -i\sigma_1\end{pmatrix},\quad \gamma^2=\begin{pmatrix}i\sigma_2 & 0\\ 0 & -i\sigma_2\end{pmatrix}.$$

At the same time we notice that since no Chern-Simons term has been introduced into the Lagrangian and we have adopted the parity-doubled gamma matrices, no photon mass will appear in our calculations. Given these assumptions, the analysis is carried out in the same manner as in the previous chapter. We start with the lowest-order Ward-Green-Takahashi (WGT) identity [96–98] for the fermions. Once again we forego the form which relates the vertex function to inverse fermion propagators,

$$k^\mu\Gamma_\mu(p,p-k)=S^{-1}(p)-S^{-1}(p-k),$$

opting instead for the form which relates the Green's function to the fermion propagators, namely

$$k^\mu S(p)\Gamma_\mu(p,p-k)S(p-k)=S(p-k)-S(p). \tag{3.4}$$

Then we use the spinor form of the Lehmann spectral representation [100–102],

$$S(p)=\int\frac{\rho(\omega)d\omega}{\not p-\omega+i\varepsilon\varepsilon(\omega)}, \tag{3.5}$$

where $\not p$ is used to denote $\gamma\cdot p$. Here we have explicitly indicated the $i\varepsilon$, which from now on will be suppressed, and so should be assumed present in all denominators. Using this spectral form, the difference of propagators in the right hand side of (3.4) takes the form

$$S(p-k)-S(p)=\int\rho(\omega)d\omega\frac{1}{\not p-\omega}\ \not k\frac{1}{\not p-\not k-\omega}. \tag{3.6}$$

Finally, since both sides of (3.4) now contain a factor of k_μ, we may now remove it to obtain the spinor form of the GT *ansatz* [40],

$$S(p)\Gamma_\mu(p,p-k)S(p-k)=\int\rho(\omega)d\omega\frac{1}{\not p-\omega}\gamma_\mu\frac{1}{\not p-\not k-\omega}\ . \tag{3.7}$$

Once again, it should be noted that this equation definitely satisfies (3.4), but is exact only up to an arbitrary transverse function, since $k_\mu T^\mu = 0$ for any transverse function T^μ, so it will have no effect on the WGT identity. Again we need to find the spectral function via the pair of Dyson-Schwinger (DS) equations [1,41–43], which in the spinor version of the theory (with bare fermion mass m_0) take the simpler form

$$
\begin{aligned}
D^{-1}_{\mu\nu}(k) &= (-\eta_{\mu\nu}k^2 + (1-\xi)k_\mu k_\nu)Z_A \\
&\qquad + ie^2 Z \int d\!\!\!{}^-{}^3 p \; \mathrm{tr}[\gamma_\mu S(p)\Gamma_\nu(p, p-k)S(p-k)] \\
&\equiv (-\eta_{\mu\nu}k^2 + (1-\xi)k_\mu k_\nu)Z_A + \Pi_{\mu\nu}(k), \qquad (3.8)
\end{aligned}
$$

for the photon, which can be represented diagramatically as shown in Figure 5.

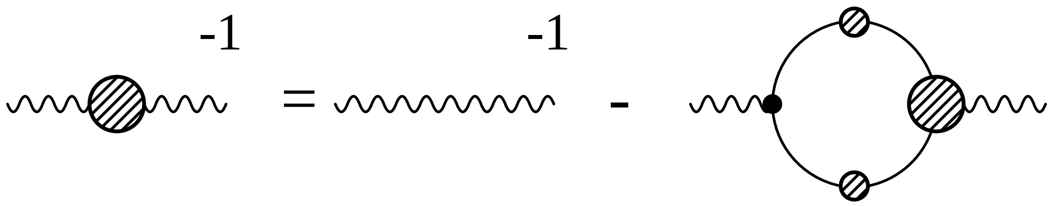


Figure 5: Photon DS equation in QED

Similarly, the fermion satisfies its own DS equation,

$$Z^{-1} = (\not{p} - m_0)S(p) - ie^2 \int d\!\!\!{}^-{}^3 k \; S(p)\Gamma_\nu(p, p-k)S(p-k)D^{\mu\nu}(k)\gamma_\mu. \qquad (3.9)$$

which is equivalent to the diagrams in Figure 6. In Figure 5 and Figure 6, wavy lines still represent photons and vertices remain as dots, but now the solid lines are introduced to signify the fermion propagators.

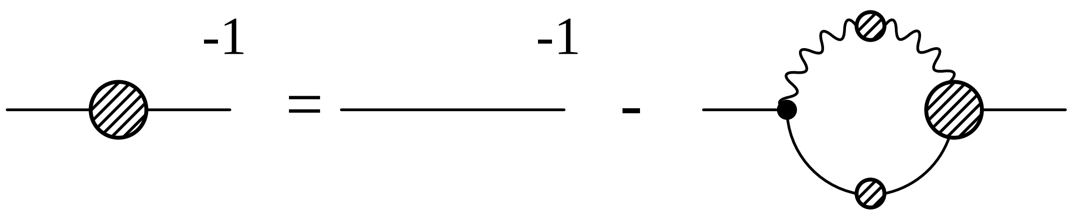


Figure 6: Fermion DS equation in QED

Equation (3.8) introduces $\Pi_{\mu\nu}$, the photon vacuum polarization, and using the *ansatz* (3.7) to obtain a linear solution of (3.8), the vacuum polarization reduces to

$$\Pi_{\mu\nu}(k) = ie^2 Z \int \rho(\omega) d\omega \int \bar{d}^3 p \ \mathrm{tr} \left[\gamma_\mu \frac{1}{\not{p} - \omega} \gamma_\nu \frac{1}{\not{p} - \not{k} - \omega} \right]. \tag{3.10}$$

Similarly, the fermion Green's function obeys

$$\begin{aligned} Z^{-1} &= (\not{p} - m_0) S(p) - \int \rho(\omega) d\omega \ ie^2 \int \bar{d}^3 k \ \frac{1}{\not{p} - \omega} \gamma_\nu \frac{1}{\not{p} - \not{k} - \omega} \gamma_\mu D^{\mu\nu}(k) \\ &= (\not{p} - m_0) S(p) + \int \frac{\rho(\omega) d\omega}{\not{p} - \omega} \Sigma(p, \omega), \end{aligned} \tag{3.11}$$

which defines $\Sigma(p, \omega)$, the fermion self-energy. This equation can be written in renormalized form, yielding

$$0 = \int \frac{\omega - m_0 + \Sigma(p, \omega)}{\not{p} - \omega} \rho(\omega) d\omega = \int \frac{\omega - m + \Sigma(p, \omega) - \Sigma(\omega, \omega)}{\not{p} - \omega} \rho(\omega) d\omega. \tag{3.12}$$

Taking the discontinuity of this equation yields

$$(\omega - m) \rho(\omega) = \int \frac{\rho(\omega') d\omega'}{\omega - \omega'} \Sigma_I(\omega, \omega'), \tag{3.13}$$

where we write $\Sigma_I(\omega, \omega') \equiv \frac{1}{\pi} \Im \Sigma(\omega, \omega')$ to represent the discontinuity of the fermion self-energy for a mass ω' fermion. Often it is convenient to expand this equation in terms of its odd and even parts, to facilitate the evaluation of the self-energy. The spectral function can only be odd or even in ω, so we write

$$\rho(\omega) = \varepsilon(\omega) [\omega \rho_1(\omega^2) + \rho_2(\omega^2)]. \tag{3.14}$$

Similarly, since Σ has no tensor indices, it can only have terms proportional to $\not{p}$ (proportional to a γ matrix), or proportional to a scalar such as ω or $\sqrt{p^2}$, so we can make the decomposition

$$\Sigma(p, \omega) = \not{p} \Sigma_1(p^2, \omega^2) + \omega \Sigma_2(p^2, \omega^2). \tag{3.15}$$

Using (3.14) and (3.15), we can then split (3.13) into the coupled pair of spinor GT equations [40, 104, 129],

$$\begin{aligned} (p^2 - m^2) \rho_1(p^2) &= \int_{m^2}^{\infty} \frac{d\omega^2}{p^2 - \omega^2} \Big[(p^2 \rho_1(\omega^2) + m \rho_2(\omega^2)) \Sigma_{1I}(p^2, \omega^2) \\ &\qquad + (\omega^2 \rho_1(\omega^2) + m \rho_2(\omega^2)) \Sigma_{2I}(p^2, \omega^2) \Big] \end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
(p^2 - m^2)\rho_2(p^2) \;=\; & \int_{m^2}^{\infty} \frac{d\omega^2}{p^2 - \omega^2} \Big[(m\rho_1(\omega^2) + \rho_2(\omega^2))p^2\Sigma_{1I}(p^2,\omega^2) \\
& + (m\omega^2\rho_1(\omega^2) + p^2\rho_2(\omega^2))\Sigma_{2I}(p^2,\omega^2)\Big]. \quad (3.17)
\end{aligned}
$$

We will use this pair of equations in the next two sections, where we will explore first the perturbative, and then the non-perturbative behaviour of the spinor theory.

3.2 Perturbation Theory

Once again, this section will be devoted to exploring QED using perturbation theory, involving a bare loop expansion in powers of e^2 in order to see if the IR perturbative behaviour of the spinor theory is more well behaved. The lowest order (e^2) perturbation correction to the photon propagator, the vacuum polarization, is shown in Figure 7 below.

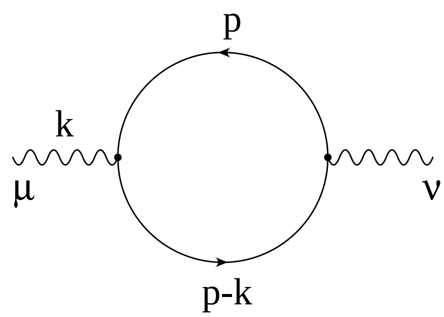


Figure 7: Photon vacuum polarization in QED

To obtain it we use (3.10), and as the starting point for the perturbation expansion we use the lowest order results for ρ_1 and ρ_2, namely

$$
\begin{aligned}
\rho_1^{(0)}(\omega^2) &= \delta(\omega^2 - m^2) \\
\rho_2^{(0)}(\omega^2) &= \omega\,\delta(\omega^2 - m^2).
\end{aligned}
$$

Using (3.14) to combine these, (3.10) yields

$$
\Pi_{\mu\nu}(k) = ie^2 \int d\!\!\,\bar{}^3 p \;\; \mathrm{tr}\left[\gamma_\mu \frac{1}{\not{p} - m}\gamma_\nu \frac{1}{\not{p} - \not{k} - m}\right]. \quad (3.18)
$$

If we write this integral in $2l$ dimensional form, then exploit the properties of products of gamma matrices, and traces of these products, which are outlined in Appendix D, we are able to write the vacuum polarization as the product of a scalar integral and the transverse projector, $-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}$, giving

$$\Pi_{\mu\nu}(k) = \frac{ie^2}{2}(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \int \bar{d}^{2l}p \, \frac{[2l\, m^2 + (2-2l)p\cdot(p-k)]}{(p^2-m^2)[(p-k)^2-m^2]}, \tag{3.19}$$

which we evaluate once again using the techniques which we associate with dimensional regularization, to obtain (once we have safely taken the limit $l \to 3/2$)

$$\Pi_{\mu\nu}(k) = \frac{e^2(\eta_{\mu\nu}k^2 - k_\mu k_\nu)}{8\pi} \left[\left(\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}} \right) \ln\left(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}} \right) - 4m \right]. \tag{3.20}$$

As in the scalar theory, this does not soften the IR behaviour of the photon propagator unless $m = 0$. Similarly, the first-order fermion self-energy is obtained by evaluating the diagram in Figure 8.

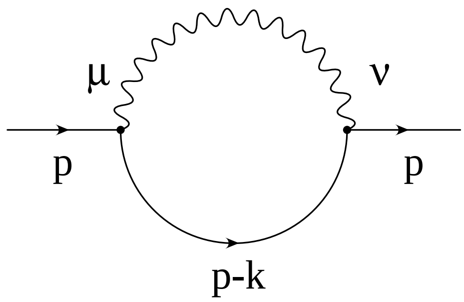


Figure 8: Contribution to fermion self-energy in QED

Since this is a perturbation calculation, we use the bare photon propagator,

$$D_{\mu\nu}(k) = -\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4}(1-\xi),$$

yielding the integral

$$\Sigma(p,m) = -ie^2 \int \bar{d}^3k \, \gamma^\mu \frac{(\not{p} - \not{k} + m)}{(p-k)^2 - m^2} \gamma^\nu \left[-\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4}(1-\xi) \right]. \tag{3.21}$$

This integral is then split, according to (3.15), into two pieces,

$$\Sigma_1(p,m) \;=\; ie^2 \int \frac{\bar{d}^3k}{k^2[(p-k)^2-m^2]} \left[\left(\frac{p\cdot k}{p^2}(2-\xi) - \xi \right) - 2(1-\xi)\frac{(p\cdot k)^2}{p^2k^2} \right]$$

and

$$\Sigma_2(p,m) \;=\; ie^2 \int \frac{\bar{d}^3k}{k^2[(p-k)^2-m^2]}(2+\xi),$$

which are evaluated to yield

$$\Sigma_1(p,m) = \frac{-e^2\xi}{8\pi}\left[\frac{m}{p^2} - \frac{(m^2+p^2)}{2(p^2)^{3/2}}\log\left(\frac{m+\sqrt{p^2}}{m-\sqrt{p^2}}\right)\right] \tag{3.22}$$

$$\Sigma_2(p,m) = \frac{-e^2(2+\xi)}{8\pi\sqrt{p^2}}\log\left(\frac{m+\sqrt{p^2}}{m-\sqrt{p^2}}\right), \tag{3.23}$$

the imaginary parts of which are found to be,

$$\Sigma_{1I}(p,m) = \frac{e^2\xi}{16\pi\sqrt{p^2}}\left(1+\frac{m^2}{p^2}\right)\theta(p^2-m^2) \tag{3.24}$$

$$\Sigma_{2I}(p,m) = -\frac{e^2}{8\pi\sqrt{p^2}}(2+\xi)\theta(p^2-m^2). \tag{3.25}$$

Now we have all we need to investigate the perturbative behaviour of the spinor theory. Once again, as in the previous chapter, we may use (3.16) and (3.17) iteratively. Substituting for Σ_{1I} and Σ_{2I}, without clothing the photon, we obtain the GT equations within this "quenched" approximation,

$$\begin{aligned}(p^2-m^2)\rho_1(p^2) = \frac{e^2}{8\pi}\int_{m^2}^{p^2}\frac{d\omega^2}{p^2-\omega^2}\frac{1}{\sqrt{p^2}}\Big[&(p^2\rho_1(\omega^2)+m\rho_2(\omega^2))\frac{\xi}{2}(1+\frac{\omega^2}{p^2})\\ &-(\omega^2\rho_1(\omega^2)+m\rho_2(\omega^2))(2+\xi)\Big]\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}(p^2-m^2)\rho_2(p^2) = \frac{e^2}{8\pi}\int_{m^2}^{p^2}\frac{d\omega^2}{p^2-\omega^2}\frac{1}{\sqrt{p^2}}\Big[&(m\rho_1(\omega^2)+\rho_2(\omega^2))\frac{\xi}{2}(p^2+\omega^2)\\ &-(m\omega^2\rho_1(\omega^2)+p^2\rho_2(\omega^2))(2+\xi)\Big].\end{aligned} \tag{3.27}$$

Notice that, as in the scalar theory, the left hand side and right hand side differ by a factor of e^2. This means that if we expand ρ_1 and ρ_2 in orders of e^2, we can introduce them at some order in the right hand side and obtain the next order on the left hand side. To obtain the first order result, we use

$$\rho_1^{(0)} = \delta(\omega^2-m^2) \qquad \text{and} \qquad \rho_2^{(0)} = \omega\ \delta(\omega^2-m^2),$$

which yields the results

$$\rho_1^{(1)}(p^2) = \frac{e^2}{8\pi\sqrt{p^2}}\left(\frac{\xi}{2p^2} - \frac{4m^2}{(p^2-m^2)^2}\right) \tag{3.28}$$

and

$$\rho_2^{(1)}(p^2) = \frac{-e^2 m}{8\pi\sqrt{p^2}} \frac{2(m^2+p^2)}{(p^2-m^2)^2} \; . \tag{3.29}$$

To the next order of approximation (e^4), we see that we are confronted (to first order in ξ) with

$$\begin{aligned}(p^2-m^2)\rho_1(p^2) \;=\; & \frac{-e^4}{64\pi^2}\int_{m^2}^{p^2} \frac{d\omega^2}{p^2-\omega^2}\frac{1}{\sqrt{p^2}\,\omega}\left[\frac{\xi(p^2+\omega^2)}{p^2(\omega^2-m^2)^2}m^2(m^2+\omega^2+2p^2)\right. \\ & \left. +\xi-(2+\xi)\frac{2m^2(m^2+3\omega^2)}{(\omega^2-m^2)^2}\right]\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}(p^2-m^2)\rho_2(p^2) \;=\; & \frac{e^4}{64\pi^2}\int_{m^2}^{p^2} \frac{d\omega^2}{p^2-\omega^2}\frac{m}{\sqrt{p^2}\,\omega}\left[\frac{2(2+\xi)(p^2m^2+p^2\omega^2+2m^2\omega^2)}{(\omega^2-m^2)^2}\right. \\ & \left. -\xi-\frac{\xi(p^2+\omega^2)(3m^2+\omega^2)}{(\omega^2-m^2)^2}\right],\end{aligned} \tag{3.31}$$

and it can easily be seen that, as in the scalar case, the above equations contain divergences at both ends of integration including the IR "cataclysm" when $\omega \to m$. Once again the perturbation expansion has been unable to obtain a finite result, since it left the photon undressed, so we consider once again non-perturbative solutions to the problem.

3.3 The Gauge Technique

We begin our non-perturbative analysis by considering an approach suggested recently [54, 133, 134]; one which modifies the non-linear approximation of Ball and Chiu [130, 131]. It is constructed as a way of bridging the gap between perturbative QCD and low-energy phenomenology, but it is also applied to QED in (2+1) dimensions [133,134]. Here they operate in the quenched approximation, and introduce a general vertex function *ansatz* containing a parameter a, which controls the magnitude of the transverse contribution, then calculate a physical (hence gauge-invariant) quantity, the fermion condensate $\langle\bar{\psi}\psi\rangle$. By varying a, they find the value at which (at least for $0 < \xi < 1$) the gauge dependence is at a minimum, which turns out to be for $a = 0.53$ [133].

The problem with the analysis as outlined by Roberts and Williams [134] stems from the criterion used to test the gauge invariance of their own and others' *ansätze*, namely

$$\int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}}\, D^{\mathrm{T}}_{\mu\nu}(p-q)\gamma_\mu S(q)\Gamma_\nu(q,p) = 0. \tag{3.32}$$

Within the quenched approximation, this condition is equivalent to demanding the propagator satisfies

$$1 = i\gamma \cdot pS(p) + \xi e^2 \int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}} \frac{i\gamma\cdot(p-q)}{(p-q)^4}\left[S(p)-S(q)\right], \tag{3.33}$$

which is equation (3.97) from [134]. We wish to consider what implications this equation (3.33) has for the theory. Perhaps the easiest way to see these effects is to consider for a moment the transformation (3.68), which is discussed in the next section, and which takes the form

$$S(x|M) = \exp(ie^2[M(x)-M(0)])S(x|0).$$

Operating on this expression with $i\,\partial\!\!\!/$ we generate

$$\begin{aligned} i\,\partial\!\!\!/ S(x|M) &= i\,\partial\!\!\!/ S(x|0)\exp(ie^2[M(x)-M(0)]) \\ &\qquad + ie^2 S(x|0)\exp(ie^2[M(x)-M(0)])\,\partial\!\!\!/ M(x) \\ &\equiv i\,\partial\!\!\!/ S(x|0)\exp(ie^2[M(x)-M(0)]) + ie^2 S(x|M)\,\partial\!\!\!/ M(x), \end{aligned} \tag{3.34}$$

then transforming to momentum space by identifying $i\,\partial\!\!\!/ \rightarrow p\!\!\!/$ we obtain

$$\begin{aligned} p\!\!\!/ S^{(M)}(p) &= -i\int dq S^{(0)}(q)\,q\!\!\!/ K(p-q) \\ &\qquad + e^2\int dq(p\!\!\!/ - q\!\!\!/)\widetilde{M}(p-q)S^{(M)}(q), \end{aligned} \tag{3.35}$$

where $K(p-q)$ is the Fourier transform of the convolution $\exp(ie^2[M(x)-M(0)])$. We know however from section 2.4 that we can write $\widetilde{M}$ explicitly as

$$\widetilde{M}(p-q) = \frac{\xi}{(p-q)^4},$$

so our relation takes the form

$$p\!\!\!/ S^{(M)}(p) = -i\int dq S^{(0)}(q)\,q\!\!\!/ K(p-q) + e^2\xi\int dq\frac{(p\!\!\!/ - q\!\!\!/)}{(p-q)^4}S^{(M)}(q). \tag{3.36}$$

We now want to compare this with (3.33) above. First notice that in (3.33), we may pull the factor of $S(p)$ out of the integral, yielding a term

$$S(p)\xi e^2 \int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}} \frac{i\gamma\cdot(p-q)}{(p-q)^4},$$

which within dimensional regularization is sure to disappear, so we may re-write (3.33) as

$$i\not{p}S(p) = 1 + i\xi e^2 \int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}} \frac{(\not{p}-\not{q})}{(p-q)^4} S(q). \tag{3.37}$$

By comparing this expression with (3.36) above, we see that for (3.37) to hold, the expression

$$\int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}} S^{(0)}(q)\ \not{\partial}K(p-q)$$

must be equal to unity. In essence, this condition is equivalent to saying that in the Landau gauge ($M = 0$), the "full" quenched fermion propagator must be equal to the bare propagator, $1/\not{p}$, which is true to order e^2, but *not* to next order [145], where the gauge at which the bare propagator becomes exact is shifted. This suggests that the claims made in [134] may only hold to order e^2.

So, to elicit our own non-perturbative solution we will need, once again within the GT, to dress the internal photon line, and this is done in precisely the same way as in section 2.3. In fact, aside from a factor of 2, the photon polarization in QED is identical to the scalar result. Since we need to weaken the $1/k^2$ singularity in the photon propagator we use the photon self energy, which comes from (3.10), once the momentum integration has been evaluated. It takes the form

$$\Pi_{\mu\nu}(k) = \frac{e^2}{8\pi}(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\int d\omega\rho(\omega)\left[\left(\sqrt{k^2} + \frac{4\omega^2}{\sqrt{k^2}}\right)\log\left(\frac{2\omega+\sqrt{k^2}}{2\omega-\sqrt{k^2}}\right) - 4\omega\right], \tag{3.38}$$

and once again we find that avoiding the IR catastrophe requires that we take a *zero mass threshold.* As a first step in this iterative process, we will assume $\rho(\omega) = \delta(\omega)$ and obtain a photon line dispersion relation

$$D_{\mu\nu}(k) = (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\frac{2c}{\pi}\int_0^\infty \frac{d\mu}{k^2-\mu^2}\frac{1}{\mu^2+c^2} - \xi\frac{k_\mu k_\nu}{k^4}$$

which is of the same form as (2.43), but now $c = e^2/8$. Using this as our dressed photon propagator we evaluate the fermion self energy, resulting in absorptive parts of the form

$$p^2\Sigma_{1I}(p^2,\omega^2) \;=\; \frac{e^2}{8\pi\sqrt{p^2}}\left[\frac{(p^2-\omega^2)^2}{\pi c}\left(\frac{1}{\sqrt{p^2}-\omega}-\frac{1}{c}\left\{\frac{\pi}{2}-\arctan(\frac{\sqrt{p^2}-\omega}{c})\right\}\right)\right.$$
$$\left.+\frac{\xi}{2}(p^2+\omega^2)-\frac{c}{\pi}(\sqrt{p^2}-\omega)+\frac{c^2}{\pi}\arctan(\frac{\sqrt{p^2}-\omega}{c})\right] \qquad (3.39)$$

$$\Sigma_{2I}(p^2,\omega^2) \;=\; -\frac{e^2}{8\pi\sqrt{p^2}}\left[\frac{4}{\pi}\arctan\left(\frac{\sqrt{p^2}-\omega}{c}\right)+\xi\right], \qquad (3.40)$$

which have then to be substituted into (3.16) and (3.17), or (3.13) to solve for the fermion spectral function. It is useful at this point to notice that if we take the perturbative limit $c\to 0$, we find that

$$\Sigma_{1I}(p,m) \;\sim\; -\frac{e^2\xi}{16\pi\sqrt{p^2}}\left(1+\frac{m^2}{p^2}\right) \qquad (3.41)$$

$$\Sigma_{2I}(p,m) \;\sim\; \frac{e^2}{8\pi\sqrt{p^2}}(2+\xi), \qquad (3.42)$$

which are exactly (3.24) and (3.25), so our solution still contains the exact perturbation result to this order. If we now recombine (3.39) and (3.40) using (3.15) to obtain

$$\Sigma_I(\omega,\omega') \;=\; \frac{e^2}{8\pi\omega}\left[\arctan\left(\frac{\omega-\omega'}{c}\right)\left\{\frac{c^2}{\pi\omega}-\frac{4\omega'}{\pi}+\frac{(\omega^2-\omega'^2)^2}{\pi\omega c^2}\right\}\right.$$
$$-\frac{c(\omega-\omega')}{\pi\omega}+\frac{\xi(\omega-\omega')^2}{2\omega}$$
$$\left.-\frac{(\omega^2-\omega'^2)^2}{2\omega c^2}+\frac{(\omega^2-\omega'^2)(\omega+\omega')}{\pi c\omega}\right], \qquad (3.43)$$

then in order to obtain the fermion spectral function we must solve

$$\omega^2\rho(\omega) \;=\; \frac{e^2}{8\pi}\int\frac{\rho(\omega')d\omega'}{\omega-\omega'}\left[\arctan\left(\frac{\omega-\omega'}{c}\right)\left\{\frac{c^2}{\pi\omega}-\frac{4\omega'}{\pi}+\frac{(\omega^2-\omega'^2)^2}{\pi\omega c^2}\right\}\right.$$
$$-\frac{c(\omega-\omega')}{\pi\omega}+\frac{\xi(\omega-\omega')^2}{2\omega}$$
$$\left.-\frac{(\omega^2-\omega'^2)^2}{2\omega c^2}+\frac{(\omega^2-\omega'^2)(\omega+\omega')}{\pi c\omega}\right]. \qquad (3.44)$$

An exact analytic solution of (3.44) is too difficult to obtain, so rather than obtaining a numerical solution we resort to exploring the behaviour of the spinor spectral equation in various asymptotic limits. At infrared momenta, i.e. $(p-m)\ll e^2$, we can once again make the approximation

$$\arctan\left(\frac{\sqrt{p^2}-m}{c}\right)\sim\frac{\sqrt{p^2}-m}{c}-\frac{(\sqrt{p^2}-m)^3}{3c^3}+\mathcal{O}\left(\frac{\sqrt{p^2}-m}{e^2}\right)^5,$$

so that the imaginary part of our self-energy behaves as

$$\Sigma_I(\omega,\omega') = \frac{(\omega-\omega')^2}{2\omega}\left(\xi + \frac{8\omega}{\pi c} - \frac{4\omega'\omega}{c^2}\right) + \mathcal{O}(\omega-\omega')^3. \tag{3.45}$$

It is important to note that if $\xi \neq 0$, the term in (3.45) proportional to ξ is the leading term, so that to leading order the self-energy behaves as

$$\Sigma_I(\omega,\omega') \simeq \frac{e^2\xi(\omega-\omega')^2}{16\pi\omega^2}, \tag{3.46}$$

resulting in a spectral function equation of the form

$$\omega\rho(\omega) \simeq \frac{e^2\xi}{16\pi\omega^2}\int_0^\omega d\omega'\rho(\omega')(\omega-\omega'). \tag{3.47}$$

Differentiating this integral equation twice we obtain the differential form,

$$\omega^2\rho''(\omega) + 6\,\omega\rho'(\omega) + \left(6 - \frac{e^2\xi}{16\pi\omega}\right)\rho(\omega) = 0, \tag{3.48}$$

the solution of which is related to a Hankel function of the second kind [146] by

$$\rho(\omega) \simeq \frac{1}{\omega^{5/2}}H_1^{(2)}\left(i\sqrt{\frac{e^2\xi}{4\pi\omega}}\right). \tag{3.49}$$

To see the approximate behaviour of this function we may then take the limiting form valid when the argument is large, finally giving us

$$\rho(\omega) \sim \frac{1}{\omega^{9/4}}\sqrt{\frac{4}{ie(\pi\xi)^{1/2}}}\exp\left(\frac{3i\pi}{4} + \sqrt{\frac{e^2\xi}{4\pi\omega}}\right). \tag{3.50}$$

If on the other hand we consider the possibility that $\xi = 0$, the leading term becomes

$$\Sigma_I(\omega,\omega') \simeq \frac{e^2(\omega-\omega')^2}{\pi^2 c\omega}, \tag{3.51}$$

and the spectral function equation becomes

$$\omega\rho(\omega) \simeq \frac{e^2}{\pi^2 c\omega}\int_0^\omega d\omega'\rho(\omega')(\omega-\omega'), \tag{3.52}$$

which has the power-law solution

$$\rho(\omega) \sim \omega^{-\frac{3}{2}\pm\frac{1}{2}\sqrt{1+8/\pi^2}}. \tag{3.53}$$

Similarly, in the ultraviolet region, $(p-m) \gg e^2$, we find that

$$\Sigma_I(\omega,\omega') \simeq \frac{e^2}{8\pi\omega}\left[\frac{\xi(\omega-\omega')^2}{2\omega} - 2\omega' + \mathcal{O}(\frac{e^2}{\omega})\right], \tag{3.54}$$

so that once again, if $\xi \neq 0$, its behaviour is dominated by the ξ term,

$$\Sigma_I(\omega,\omega') \simeq \frac{e^2\xi(\omega-\omega')^2}{16\pi\omega^2}, \tag{3.55}$$

resulting in a spectral function equation of the form

$$\omega\rho(\omega) \simeq \frac{e^2\xi}{16\pi\omega^2}\int_0^\omega d\omega'\rho(\omega')(\omega-\omega'), \tag{3.56}$$

exactly as in the IR case. Its solution is still related to a Hankel function of the second kind [146] by

$$\rho(\omega) \simeq \frac{1}{\omega^{5/2}}H_1^{(2)}\left(i\sqrt{\frac{e^2\xi}{4\pi\omega}}\right), \tag{3.57}$$

but for UV momenta the argument of the Hankel function is small, so its approximate behaviour is given by taking the small argument limiting form, giving us

$$\rho(\omega) \sim \frac{1}{\omega^2}\sqrt{\frac{16\pi}{e^2\xi}}. \tag{3.58}$$

Finally, if we consider the case that $\xi = 0$ in the UV limit, we find that the fermion behaviour is dominated by

$$\Sigma_I(\omega,\omega') \simeq \frac{-e^2\omega'}{4\pi\omega}, \tag{3.59}$$

giving an integral equation of the form

$$\omega\rho(\omega) \simeq \frac{-e^2}{4\pi\omega^2}\int^\omega \rho(\omega')d\omega'\omega'. \tag{3.60}$$

A solution of this equation is

$$\rho(\omega) \simeq \frac{e^4}{32\pi^2\omega^3}\exp\left(\frac{e^2}{4\pi\omega}\right), \tag{3.61}$$

where we have adjusted the (arbitrary) normalization constant so that by taking the limit $e^2 \to 0$, that is, by returning to the perturbation approximation, we ensure that $\rho(\omega) \to \delta(\omega)$ and $S(p) \to 1/\not{p}$. So, once again we have managed to obtain a non-perturbative result which contains the order e^2 perturbation result, but is free of IR singularities and whose behaviour can be obtained at asymptotic limits.

It would be nice to see how (3.61) compares with the results of others, which we can do by evaluating the fermion propagator, which (in the UV limit) gives

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} \frac{\rho(t)\varepsilon(t)dt}{\omega - t} \\ &= \frac{1}{\omega} - \frac{e^4}{16\pi^2\omega^3} \exp\left(\frac{e^2}{4\pi\omega}\right) Ei\left(\frac{-e^2}{4\pi\omega}\right), \end{aligned} \tag{3.62}$$

where $Ei(x)$ is the exponential integral. For small (negative) argument, as we have here, Ei behaves as [146]

$$Ei(-x) \simeq \gamma + i\pi + \log(x),$$

where γ is Euler's constant. Also, for large ω, $\exp\left(\frac{e^2}{4\pi\omega}\right) \to 1$, so $S(p)$ takes the form

$$S(\omega) \simeq \frac{1}{\omega} - \frac{e^4}{16\pi^2\omega^3} \log\left(\frac{e^2}{4\pi\omega}\right). \tag{3.63}$$

Let us compare the behaviour of (3.63) with the results of Guendelman and Radulovic [47]. They obtain an expression for the $(2n)$th term in the expansion of the fermion self-energy to be

$$\Sigma^{(2n)} \simeq pA^{(2n)} \frac{e^{4n}}{p^{2n}} \log^n(\frac{e^2}{p}),$$

where the $A^{(2n)}$ are constants. Since higher order terms become less leading in behaviour, we consider the dominant first term ($\Sigma^{(1)} = 0$ in the Landau gauge),

$$\Sigma^{(2)} \simeq \frac{-e^4}{48\pi^2 p} \log(\frac{e^2}{p}),$$

so that if we expand the propagator binomially,

$$\begin{aligned} S(p) &= \frac{1}{\not{p} - \Sigma} \\ &\simeq \frac{1}{\not{p}} - \frac{e^4}{48\pi^2 p^3} \log(\frac{e^2}{p}), \end{aligned} \tag{3.64}$$

plus higher-order terms in $\log(\frac{e^2}{p})$. This equation is of the same form as (3.63), differing only by a factor of 4 in the second term.

We may also look at the asymptotic behaviour of the fermion propagator found in Ref. [37], which has the form

$$S(\omega) = \frac{1}{\omega}\left[1 + \frac{\sqrt{\pi}}{Z(\omega)}\exp[Z^2(\omega)]\,\mathrm{erfc}[Z(\omega)] - \sqrt{\pi}Z(\omega)\exp[-Z^2(\omega)]\right], \quad (3.65)$$

where

$$Z(\omega) = \frac{\omega}{\sqrt{2}(e^4/48\pi^2)\log(\omega/e^2)}.$$

For $\omega \to \infty$, $Z(\omega)$ also approaches ∞, so we may use the approximation [146]

$$\begin{aligned}\sqrt{\pi}Ze^{Z^2}\mathrm{erfc}(Z) &= 1 + \sum_{m=1}^{\infty}(-1)^m\frac{1.3\ldots(2m-1)}{(2Z^2)^m} \\ &\simeq 1 + \mathcal{O}(1/Z^2), \end{aligned} \quad (3.66)$$

and since the term of the form $Z/\exp(Z^2)$ in (3.65) will be dominated by the term containing $1/Z^2$, we obtain the approximation

$$S(\omega) \sim \frac{1}{\omega} + \frac{2(\frac{e^4}{48\pi^2})^2\log^2(\omega/e^2)}{\omega^3}. \quad (3.67)$$

This result does not agree with those of Guendelman and Radulovic [47] above or the present work. It is correct to within a factor of $e^4\log(\omega/e^2)$, but the second term in (3.67) is of the same order as the next contribution in Ref. [47] and the present work, suggesting that their approach misses some of the asymptotic behaviour of S.

3.4 Gauge Covariance Relations

In order to comprehend the gauge dependence of the spectral functions away from threshold, we will once again examine their gauge covariance relations. As in the scalar theory, we use gauge transformation properties [140–144], which link the variation in the photon propagator given by

$$D_{\mu\nu}^{(M)}(k) = D_{\mu\nu}^{(0)}(k) + k_\mu k_\nu M(k)$$

to that of the fermion propagator, namely

$$\begin{aligned} S(x|K) &= \exp(ie^2[M(x) - M(0)])S(x|0) \\ &= \exp(-iK\sqrt{x^2})S(x|0), \end{aligned} \quad (3.68)$$

where $K = e^2\xi/8\pi$ and $M(x) = -K\sqrt{x^2}$. In obtaining the gauge behaviour of the fermion spectral function, we once again follow references [106,107] and begin by using the Lehmann spectral representation in position space,

$$S(x|K) = \int \rho(\omega|K) S_c(x,\omega) d\omega, \tag{3.69}$$

to rewrite (3.68) as

$$S(x|K) = \exp(-iK\sqrt{x^2}) \int \rho(\omega|0) S_c(x,\omega) d\omega. \tag{3.70}$$

Using the fact that the free fermion propagator (mass ω) equals

$$S_c(x,\omega) = -(i\,\not\partial + \omega)\frac{e^{-\omega\sqrt{-x^2}}}{4\pi\sqrt{x^2}}\ , \tag{3.71}$$

and since $S(p)$ has a Lehmann spectral representation (in any gauge K),

$$S(p|K) = \int \rho(\omega|K) S_c(p,\omega) d\omega, \tag{3.72}$$

we can take the Fourier transform of (3.70), giving

$$\begin{aligned}\int \rho(\omega|K)\frac{d\omega}{\not p - \omega} &= -\int \rho(\omega|0)d\omega \int d^3x\ e^{ip\cdot x} e^{-iK\sqrt{x^2}}(i\,\not\partial + \omega)\left(\frac{e^{-\omega\sqrt{-x^2}}}{4\pi\sqrt{x^2}}\right) \\ &= \frac{1}{p^2}\int \rho(\omega|0)d\omega\left[\frac{p^2}{\not p - (\omega+K)} - \frac{\not p K}{\not p - (\omega+K)}\right. \\ &\qquad\left. - \frac{\not p K}{2\sqrt{p^2}}\ \log\left(\frac{K+\omega-i\sqrt{-p^2}}{K+\omega+i\sqrt{-p^2}}\right)\right],\end{aligned} \tag{3.73}$$

where we have evaluated the x-integration by making a Euclidean rotation. Those terms proportional to $\not p$ can be combined and written as a dispersion relation, finally giving the general covariance relation,

$$\int\rho(\omega|K)d\omega\frac{1}{\not p - \omega} = \int\rho(\omega|0)d\omega\left[\frac{1}{\not p - (\omega+K)} - \int_{\omega+K}^{\infty}\frac{2K\,d\mu}{(p^2-\mu^2)(\not p + \mu)}\right]. \tag{3.74}$$

Rationalizing this equation, by splitting $\rho(\omega)$ into odd and even parts

$$\rho(\omega) = \varepsilon(\omega)[\omega\rho_1(\omega^2) + \rho_2(\omega^2)]$$

and separating odd and even terms in $\not p$, we get to first order in K:

$$\begin{aligned}\int\frac{d\omega}{p^2-\omega^2}&\Big[\varepsilon(\omega)\omega\Big(\rho_1(\omega^2|K) - \rho_1((\omega-K)^2|0) - 2K\rho_2'(\omega^2|0)\Big) - 2K\delta(\omega)\rho_2(\omega^2|0)\Big] \\ &= -2K\int d\omega\rho_2(\omega^2|0)\varepsilon(\omega)\int_{\omega+K}^{\infty}\frac{d\mu}{(p^2-\mu^2)^2}\end{aligned} \tag{3.75}$$

and

$$\int \frac{d\omega}{p^2-\omega^2}\left[\varepsilon(\omega)\omega(\rho_2(\omega^2|K)-\rho_2((\omega-K)^2|0)-2K\omega^2\delta(\omega)\rho_1(\omega^2|0)\right]$$
$$= -2K\int \frac{d\omega^2\rho_1(\omega^2|0)}{p^2-(\omega+K)^2}\theta(p^2-(\omega+K)^2); \tag{3.76}$$

whereupon, taking the discontinuity of these equations we obtain

$$-\rho_1(p^2|K)+\rho_1((p-K)^2|0)+2K\rho_2'(p^2|0)$$
$$= -2K\int_{-\infty}^{\infty} d\omega\rho_2(\omega^2|0)\left[\frac{1}{4p^3}\theta(p^2-\omega^2)\right]\varepsilon(\omega) \tag{3.77}$$

and

$$-\rho_2(p^2|K)+\rho_2((p-K)^2|0) = K\rho_1(p^2|0)/2. \tag{3.78}$$

We may now use these equations to check the consistency of our perturbation theory results (3.28), (3.29) which when written to order e^2 take the form

$$\rho_1(p^2) = \delta(p^2-m^2)-\frac{e^2}{8\pi\sqrt{p^2}}\left(\frac{\xi}{2p^2}-\frac{4m^2}{(p^2-m^2)^2}\right) \tag{3.79}$$
$$\rho_2(p^2) = p\,\delta(p^2-m^2)+\frac{e^2m}{8\pi\sqrt{p^2}}\frac{2(m^2+p^2)}{(p^2-m^2)^2}\,. \tag{3.80}$$

Substitution of these spectral relations into our covariance relations shows that the results reassuringly satisfy them for all ξ. Likewise, we may expand our GT results (in any asymptotic region) in powers of e^2, and to order e^2 we similarly find that the covariance relations are satisfied.

Chapter 4

Chern-Simons Field Theory

In this chapter, we will allow for the first time the presence of a parity-violating photon mass in our theory, taking the form of a Chern-Simons (CS) term in the Lagrangian. We begin by outlining the elements of these CS theories, and the specific problems associated with performing calculations in such theories. We will then go on to discuss how we overcome the problem of regularization in this theory, then perform lowest order perturbation theory to see if the IR problems which plague SED and QED occur here as well.

4.1 Background

As we outlined in the Introduction, there has been much recent interest in CS theories in three dimensions, due to their unusual behaviour and their applicability to such diverse areas of physics.

The pure CS theory, studied first by Witten [64] and consisting of only the term

$$\epsilon_{\mu\nu\lambda}A^{\lambda}F^{\mu\nu},$$

was found to be exactly soluble [64,65]. The observables of this theory are Wilson lines, which are gauge-invariant and allow the action to remain independent of the metric [64]. The vacuum expectation values of the Wilson lines were found to correspond [64, 68–71] to the link invariants in the Jones theory [66, 67] and its generalizations. Also, quite unexpectedly, Witten observed that CS theories in 3D are intimately related to rational conformal field theories (CFT's) in 2D. This connection can be understood by studying the CS theory on a manifold of the form $\Sigma \times R$, where the non-compact direction R is interpreted as time and can be quantized, and the Σ is some 2D space. The 2D CFT is then recovered in two related ways. First, if Σ is compact, the states of the Hilbert space $\mathcal{H}_\Sigma$ correspond to the conformal blocks of the rational CFT [64,72–74]. Alternatively, by quantizing the theory on a space with a boundary, the Hilbert space becomes infinite dimensional, and corresponds to a representation of the chiral algebra of the 2D rational CFT [64,75,76]. This has been extended so that one can organize all known rational CFT's by choosing the appropriate gauge group and coupling constants [76].

By coupling CS theory to sources, we can generate particles with generalized statistics, known as anyons [29, 30, 32, 33]. Such particles were precisely what was needed in explaining the fractional quantum Hall effect, where there exist excitations displaying fractional statistics [77–79]. It is also believed that anyons may play a role in explaining the magnetic properties of layered copper-oxide compounds, which are known to display high T_c superconductivity [80–82]. CS gauge theory [83,84] has been shown to precisely describe the quantum mechanics of anyons.

We will begin our study of CS theory by considering normal QED, as studied in chapter 3, but now use the 2×2 form of the gamma matrices, since parity will be violated in this theory anyway [37,45,46]. If we perform perturbative calculations within this theory, we find that the presence of a fermion mass causes the photon to dynamically acquire a topological mass term, suggesting that we should include such a CS term directly into the Lagrangian. This extra gauge-invariant term results in our Lagrangian taking the form

$$\mathcal{L} = \bar{\psi}(i\,\partial\!\!\!/ - m - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4\xi}(\partial_\mu A^\mu)^2 + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^\lambda,$$

which is identical to that given in equation (3.1), apart from the extra CS contribution. As usual, the Feynman rules are obtained by taking functional derivatives of the Lagrangian with respect to the relevant fields, and the most interesting result here is that the photon propagator (once inverted) takes the form

$$D_{\mu\nu} = \frac{-1}{k^2 - \mu^2}\left[\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{i\,\mu}{k^2}\epsilon_{\mu\nu\lambda}k^\lambda\right] - \frac{\xi k_\mu k_\nu}{k^4}, \tag{4.1}$$

which behaves (when $\xi = 0$) in the IR sector, $k \to 0$, as $1/k$. The full set of Feynman rules are given in Appendix A. Given these Feynman rules, the next step is to calculate Feynman diagrams, and when we attempt to do so we find (not surprisingly) superficial UV divergences.

An important consideration when studying any theory is the method employed to regularize these UV divergent integrals. Several attempts have been made to successfully regularize CS theory. Pauli-Villars regularization has, for instance, been applied to the theory [60]. This is where the UV divergence of a momentum integral is handled by introducing a mass-like regulator M which has the effect

$$\int \frac{đ^3 p}{p^2 - m^2} \to \int đ^3 p\left(\frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2}\right) \equiv \int đ^3 p \int_{M^2}^{m^2} d\mu^2 \frac{1}{(p^2 - \mu^2)^2}, \tag{4.2}$$

rendering the apparently divergent integral into a tenable form. Setting $M \to \infty$ after doing the momentum integral would return the required result. The problem with this approach is that it has no physical interpretation, it is no good in a massless theory, and it is not in general applicable to a non-abelian theory. Another technique is analytic regularization [85, 86]. This involves analytically continuing the exponent of the propagators so that

$$\int \frac{đ^3 p}{p^2 - m^2} \to \int \frac{đ^3 p}{(p^2 - m^2)^{1+\epsilon}}, \tag{4.3}$$

which once again allows the momentum integration to be performed. Here, on the other hand, the problem is that the gauge symmetry of the theory is broken. Once again, the form of the denominator has no physical interpretation when analytically continued away from $\epsilon = 0$, and the new Feynman rule for the fermion propagator complicates the evaluation of the integral. The problem has also been considered [87] using nonlocal regularization [147]. This relatively new method involves altering the interaction part of the classical action by "smearing" the fields, using an operator which involves derivatives of the fields. In the limit this method returns the correct results, but its disadvantage is that it complicates the Feynman rules of the theory by introducing a four-point interaction and a fermion measure factor. Finally, dimensional regularization has been applied to Chern-Simons theory [88–92], using methods we will discuss below.

The various techniques described above have not yielded a conclusive result. We wish to consider the calculation using dimensional regularization, as it avoids the UV divergences in a very natural and unobtrusive way, without complicating the Feynman rules, and allows simple access to higher orders of perturbation theory. Once we have obtained our result, we will discuss the reasons for the apparent disparity in the results of others.

4.2 Dimensional Regularization

Dimensional regularization involves setting up the field theory in arbitrary n-dimensional space-time. This is done to exploit the fact that by analytically continuing away from physical dimensions, the momentum integration becomes tenable, and any persistent divergence is seen as a pole in gamma functions. A problem arises however whenever a theory involves objects whose properties depend explicitly on the dimension, such as γ_5 matrices in even dimensions, and ϵ tensors in odd-dimensional theories such as the one we are now considering. In (3+1)D, this problem has been overcome by replacing the γ_5 by the fourfold antisymmetric product of the other gamma matrices [148], and axial vectors by threefold antisymmetric tensors

$$\gamma_5 \to \Gamma_{[\mu}\Gamma_\nu\Gamma_\sigma\Gamma_{\tau]}, \qquad \gamma_\alpha\gamma_5 \to \Gamma_{[\mu}\Gamma_\nu\Gamma_{\sigma]},$$

where the Γ's are just the $2l$-dimensional generalization of the normal γ matrices, so that μ, ν, σ, and τ run from 0 to $2l-1$.

In (2+1)D, things are not quite as simple. Even if no ϵ term is included in the Lagrangian, it has the potential to be generated dynamically, so in a dimensional context the ϵ tensor must be generalized to exist within $2l+1$ dimensions. It is not good enough to leave it as $\epsilon_{\mu\nu\lambda}$ and continue to $2+1+\varepsilon$ dimensions as several authors have done [89–92]. This technique involves purportedly going to arbitrary D dimensions, but retaining the normal $\epsilon_{\mu\nu\lambda}$ tensor, which is defined to be "essentially" 3 dimensional. This is achieved by defining

$$g_{\mu\nu} = \hat{g}_{\mu\nu} \bigoplus \tilde{g}_{\mu\nu}$$

to be the D-dimensional metric, where $\tilde{g}_{\mu\nu}$ is the projection of $g_{\mu\nu}$ onto the 3 dimensional Minkowski space and $\hat{g}_{\mu\nu}$ is the projection onto the orthogonal $D-3$ space. If u defines a D dimensional vector, and $u_\mu = \hat{u}_\mu \oplus \tilde{u}_\mu$, then

$$\epsilon^{\mu_1\mu_2\mu_3}\epsilon_{\nu_1\nu_2\nu_3} = \tilde{g}^{\mu_1}_{[\nu_1}\tilde{g}^{\mu_2}_{\nu_2}\tilde{g}^{\mu_3}_{\nu_3]} \tag{4.4}$$

and

$$\epsilon^{\mu_1\mu_2\mu_3}\hat{u}_{\mu_3} = 0. \tag{4.5}$$

The problem is that this method leads to a host of complicated structures, splitting all tensorial objects into their projections onto the two spaces, only at the end returning to the simple results we expect. The technique works to one loop, but we would expect it to at that order since the term proportional to ϵ does not need to be regularized, as can be seen by power counting.

The problem is that if we wish instead for the ϵ tensor to become truly $(2l+1)$-dimensional, it must have $2l+1$ indices, only three of which would be used by the usual AF, so must decide what to do with the extra indices. Several methods to "soak up" these extra indices suggest themselves. We could write the Chern-Simons contribution to the Lagrangian as

$$\epsilon_{\mu_1\mu_2\mu_3\cdots\mu_{2l}\mu_{2l+1}}A^{\mu_1}F^{\mu_2\mu_3}\ldots F^{\mu_{2l}\mu_{2l+1}},$$

retaining A^μ as the natural gauge potential and $F^{\mu\nu}$ as its associated field strength. This would indeed be a genuine $(2l+1)$-dimensional term, yielding the correct (2+1)D limit. Unfortunately this approach requires the consideration of new processes and diagrams, since each F corresponds to an extra photon line. The effect of this can be seen most easily by considering a few examples. If we were attempting to calculate the one loop vacuum polarization of the photon, and attempted to regularize using this scheme, we would be faced with two diagrams which contribute in $2l+1$ dimensions, shown in Figure 9. In the first diagram,

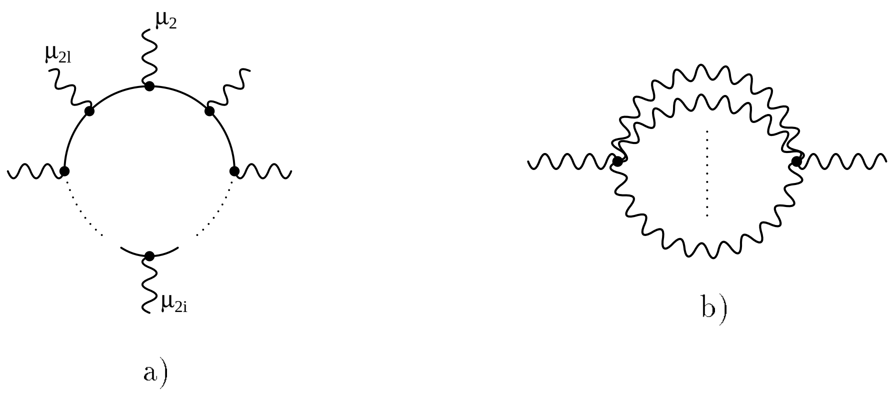


Figure 9: Contributions to the vacuum polarization.

Figure 9a, the complication is only moderate, involving integration over a loop with l external legs. In Figure 9b however we are faced with a far more challenging task, having to integrate over $l-1$ internal loops. Similarly, the fermion self-energy would receive a contribution from Figure 10 below, which also involves

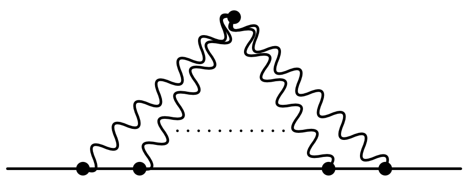


Figure 10: Contribution to the fermion self-energy.

integration over an arbitrary number of momentum loops.

Calculations within this framework become extremely complicated, even when only considering processes to one loop in the original theory, and would be akin to using a γ_5 which is the antisymmetric product of *all* γ matrices for treating the chiral anomaly. (Although, we will be forced to consider such calculations when we study higher-odd-dimensional theories in the next chapter).

Instead we will depart from the normal approach and regard the "gauge" field as an l-component antisymmetric tensor [93]. (Of course, in a nonabelian theory we would encounter extra gauge degrees of freedom, necessitating consideration of a nest of ghost-for-ghost mechanisms.) Now the Chern-Simons term retains the *bilinear* form

$$\mathcal{L}_{\rm CS} = \epsilon_{\mu_1\cdots\mu_{2l+1}} A^{\mu_1\cdots\mu_l} F^{\mu_{l+1}\cdots\mu_{2l+1}}, \tag{4.6}$$

where

$$F^{\mu_{l+1}\cdots\mu_{2l+1}} = \partial^{[\mu_{l+1}} A^{\mu_{l+2}\cdots\mu_{2l+1}]}$$

is the antisymmetric curl of $A^{\mu_1\cdots\mu_l}$. Our Chern-Simons term has the advantage that even in $2l+1$ dimensions, the processes of the theory are unchanged. Also, this formalism, adopted within dimensional regularization, continues to work in topological theories, in contrast to the conventional approach. It is not obvious however what the physical significance of the $A^{\mu_1\cdots\mu_l}$ is, particularly as it couples to a non-conserved tensor current $\overline{\psi}\gamma_{[\mu_1}\cdots\gamma_{\mu_l]}\psi$, but it is not crucial to visualize it for arbitrary l since it reverts to a *bona fide* gauge field in $l=1$, i.e. in three dimensions.

We therefore generalize the Chern-Simons (CS) Lagrangian in (2+1)D,

$$\mathcal{L}_{(2+1)\rm D} = \bar{\psi}(i\,\partial\!\!\!/ - m - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4\xi}(\partial_\mu A^\mu)^2 + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^\lambda, \tag{4.7}$$

to arbitrary $(2l+1)$ dimensions by making the extension described above, which yields

$$\begin{aligned}\mathcal{L} \;=\; & \frac{(-1)^l}{2(l+1)!}\, F^{\lambda_1\cdots\lambda_{l+1}}F_{\lambda_1\cdots\lambda_{l+1}} + \frac{\mu}{2(l+1)!\,(l!)^2}\,\epsilon_{\lambda_1\cdots\lambda_{2l+1}}F^{\lambda_1\cdots\lambda_{l+1}}A^{\lambda_{l+2}\cdots\lambda_{2l+1}} \\ & -\frac{(-1)^l}{2\xi}(\partial_\sigma A^{\sigma\nu_1\cdots\nu_{l-1}})^2 + \bar{\psi}(i\,\partial\!\!\!/ - m - e\gamma^{[\mu_1\cdots\mu_l]}A_{\mu_1\cdots\mu_l})\psi,\end{aligned} \tag{4.8}$$

where $\gamma_{[\mu_1\cdots\mu_l]} \equiv \gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_l]}/l!$ is the antisymmetric product of l γ_{μ_i}'s, which themselves are just the $2l$ non-parity-doubled $(2^l\times 2^l)$ γ-matrices, as well as the full "γ_5", which acts as the last one, γ_{2l+1}, as outlined in Appendix D. Taking functional derivatives with respect to $A^{\mu_1\cdots\mu_l}$ and $A^{\nu_1\cdots\nu_l}$, we obtain the inverse propagator for this tensor field

$$\begin{aligned}(D^{-1})_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k) \;\equiv\; & \frac{\delta^2\mathcal{L}}{\delta A^{\mu_1\cdots\mu_l}\delta A^{\mu_1\cdots\mu_l}} \\ =\; & (-1)^l\left(k^2\eta_{\mu_1[\nu_1}\cdots\eta_{\mu_l\nu_l]} - (1-\frac{1}{\xi})k_{[\mu_1}\eta_{\mu_2\sqsubset\nu_2}\cdots\eta_{\mu_l]\nu_l}k_{\nu_1\sqsupset}\right) \\ & +i\mu\, k^\sigma\epsilon_{\sigma\mu_1\cdots\mu_l\nu_1\cdots\nu_l},\end{aligned} \tag{4.9}$$

where the [] in $\eta_{\mu_1[\nu_1} \cdots \eta_{\mu_l\nu_l]}$ denote antisymmetry in the ν_i's and the [] and $\sqsubset\sqsupset$ in $k_{[\mu_1}\eta_{\mu_2\sqsubset\nu_2} \cdots \eta_{\mu_l]\nu_l}k_{\nu_1\sqsupset}$ denote antisymmetry in the μ_i's and ν_i's respectively. Operations such as inverting this generalized propagator have become more complicated due to the extra tensor structure involved, but it is still possible, yielding

$$\begin{aligned} D_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k) &= \frac{(-1)^l}{l!\,(k^2+(-1)^l\mu^2)}\left(\eta_{\mu_1[\nu_1}\cdots\eta_{\mu_l\nu_l]} - \frac{1}{k^2}k_{[\mu_1}\eta_{\mu_2\sqsubset\nu_2}\cdots\eta_{\mu_l]\nu_l}k_{\nu_1\sqsupset}\right) \\ &\quad -\frac{i\mu}{l!\,k^2(k^2+(-1)^l\mu^2)}k^\sigma\epsilon_{\sigma\mu_1\cdots\mu_l\nu_1\cdots\nu_l} \\ &\quad +\frac{(-1)^l\xi}{l!\,k^4}k_{[\mu_1}\eta_{\mu_2\sqsubset\nu_2}\cdots\eta_{\mu_l]\nu_l}k_{\nu_1\sqsupset}. \end{aligned} \tag{4.10}$$

This will reduce to the normal (2+1)D gauge propagator, $D_{\mu\nu}$ by collapsing to $l=1$, yielding the standard result [60, 61]. It should be noted that if l is even, the photon propagator becomes tachyonic. This would be a serious problem, but it can be seen that for even l, (4.6) becomes a total divergence and will not contribute to the theory. Similarly we obtain from (4.8) a 'gauge'-fermion-fermion vertex which takes the form

$$-\,e\gamma^{[\mu_1\cdots\mu_l]}, \tag{4.11}$$

and is not conserved unless $l=1$.

Having set up this system for extending CS theory to arbitrary $2l+1$ dimensions, we may now test its suitability to dimensional regularization, by considering some perturbation calculations.

4.3 Perturbation Theory

In this section we will investigate this generalized, arbitrary-dimensional version of abelian CS theory using dimensional regularization, and use the results obtained to investigate the IR behaviour of CS theory within a perturbative framework. Using the usual free fermion propagator, we may now calculate the vacuum polarization, the one-loop correction to the bare "gauge" propagator,

$$(\mathcal{D})^{-1}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} = (D)^{-1}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} + \Pi_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} \tag{4.12}$$

$$\Pi_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k) = (-1)^{[\frac{l}{2}]} i e^2 \int \frac{d^{2l+1}p}{(2\pi)^{2l+1}} \mathrm{tr}\Big[\gamma_{[\mu_1\cdots\mu_l]} S(p+k) \gamma_{[\nu_1\cdots\nu_l]} S(p)\Big]. \quad (4.13)$$

Here $[\frac{l}{2}]$ is just the integer part of $\frac{l}{2}$. Using the methods associated with dimensional regularization, including introduction of a Feynman parameter, and evaluating the $(2l+1)$-dimensional momentum integral, we find an expression for the "vacuum polarization",

$$\Pi_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k) = \Pi_1(k^2) T^{(1)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} + \Pi_2(k^2) T^{(2)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} + \Pi_3(k^2) T^{(3)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}. \quad (4.14)$$

Here,

$$\begin{aligned}
T^{(1)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} &= \Big(k_{[\nu_1}\eta_{\nu_2\sqsubset\mu_2}\cdots\eta_{\nu_l]\mu_l}k_{\mu_1\sqsupset} - k^2\eta_{\mu_1[\nu_1}\cdots\eta_{\mu_l\nu_l]}\Big) \\
T^{(2)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} &= i\, m k^\sigma \epsilon_{\sigma\mu_1\cdots\mu_l\nu_1\cdots\nu_l} \\
T^{(3)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l} &= \eta_{\mu_1[\nu_1}\cdots\eta_{\mu_l\nu_l]}
\end{aligned}$$

and

$$\begin{aligned}
\Pi_1(k^2) &= \frac{e^2(-2)^{l+1}\Gamma(\frac{3}{2}-l)}{(4\pi)^{l+\frac{1}{2}}} \int_0^1 \frac{d\alpha\,\alpha(1-\alpha)}{(m^2-k^2\alpha(1-\alpha))^{\frac{3}{2}-l}} \\
\Pi_2(k^2) &= \frac{e^2(-1)^{[\frac{l}{2}]}2^l\Gamma(\frac{3}{2}-l)}{(4\pi)^{l+\frac{1}{2}}} \int_0^1 \frac{d\alpha\Big((-1)^l(1-\alpha)-\alpha\Big)}{(m^2-k^2\alpha(1-\alpha))^{\frac{3}{2}-l}} \\
\Pi_3(k^2) &= \frac{e^2(-2)^l\Gamma(\frac{1}{2}-l)}{(4\pi)^{l+\frac{1}{2}}} \int_0^1 d\alpha \left(\frac{(1-l)}{(m^2-k^2\alpha(1-\alpha))^{\frac{1}{2}-l}} - \frac{m^2(\frac{1}{2}-l)(1+(-1)^l)}{(m^2-k^2\alpha(1-\alpha))^{\frac{3}{2}-l}}\right).
\end{aligned}$$

We would like $\Pi_{\mu\nu}$ to satisfy the Ward identity for the vacuum polarization, $k^\mu\Pi_{\mu\nu} = 0$, when $l = 1$. We would also have liked our $\Pi_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ to obey this relation, but that cannot be for arbitrary l, since the current to which A couples is not conserved except for $l = 1$. In (4.14) above, $k^{\mu_1}T^{(1)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ and $k^{\mu_1}T^{(2)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ are always equal to zero, but $k^{\mu_1}T^{(3)}_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ persists and $\Pi_3 = 0$ only if $l = 1$. This can be summarized by saying that effectively $k^{\mu_1}\Pi_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ contains an evanescent factor of $(l-1)$. The important point is that in odd dimensions, since Π_1, Π_2 and Π_3 do *not* contain $1/(l-1)$ divergences[a], the Ward Identity is always satisfied in the limit, and the theory is free of anomalies. It should also be noted that if l is even, Π_2 contains a factor of $(1-2\alpha)$, which causes the ϵ contribution to $\Pi_{\mu\nu}$ to disappear. This is to be expected, since $\mathcal{L}_{\mathrm{CS}}$ becomes a total divergence for even l, as we discussed earlier.

Since the Ward identity becomes satisfied, we can indeed calculate the gauge-invariant 3D vacuum polarization by letting $l \to 1$ without fear. After performing the α integration, $\Pi_{\mu\nu}$ reduces to

$$\Pi_{\mu\nu}(k) = \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi_1(k^2) + im\epsilon_{\mu\nu\sigma}k^\sigma\Pi_2(k^2) \tag{4.15}$$

where

$$\begin{aligned}\Pi_1(k^2) &= \frac{e^2}{16\pi}\left[\left(\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}}\right)\ln\left(\frac{2m+\sqrt{k^2}}{2m-\sqrt{k^2}}\right) - 4m\right] \\ \Pi_2(k^2) &= \frac{e^2}{4\pi\sqrt{k^2}}\ln\left(\frac{2m+\sqrt{k^2}}{2m-\sqrt{k^2}}\right).\end{aligned}$$

Studying the asymptotic behaviour of $\Pi_{\mu\nu}$, we find that as $k \to 0$, and provided $m \neq 0$,

$$\Pi_1(k^2) = \frac{e^2k^2}{12\pi m}, \qquad \Pi_2(k^2) = \frac{e^2}{4\pi m}. \tag{4.16}$$

We also note that if $m = 0$

$$\Pi_1(k^2) = -\frac{e^2\sqrt{-k^2}}{16}, \qquad \Pi_2(k^2) = \frac{e^2}{4\sqrt{-k^2}}, \tag{4.17}$$

but since the coefficient of Π_2 contains a factor of m, the Π_2 contribution disappears, and we are left with the equivalent of the usual parity-doubled vacuum polarization [37].

To place these results in perspective we will now look briefly at the asymptotic behaviour of the $\Pi_{\mu\nu}$ obtained by others. Firstly, Stam's [56] expressions for Π_1 and Π_2 correspond exactly to ours. The $\Pi_{\mu\nu}$ obtained by Pimentel, Suzuki

[a]Just such a divergence leads to the axial Adler-Bell-Jackiw anomaly in 4D chiral gauge theory.

and Tomazelli [85], using analytic regularization, also shows exactly the same behaviour as (4.16) and (4.17) above provided that the limiting procedure is properly undertaken[b]. Hand and Moffat similarly obtain the correct limiting behaviour for Π, using analytic regularization [86] and nonlocal regularization [87]. The results of Appelquist, Bowick, Karabali and Wijewardhana are also of precisely the same form as the present work [149]. Finally, comparing with the work of Deser, Jackiw and Templeton [60], who used Pauli-Villars regularization, we find some interesting differences. Again Π_1 shows the usual behaviour, but their ϵ contribution does *not* disappear for $m = 0$. It has been suggested that this is a product of the Pauli-Villars regularization scheme, but it is our belief that it arises due to their method of expanding Π_1 and Π_2 around $p^2 = 0$, and discarding the $\Pi_2(0)$ term—believed to be zero—which in actuality cancels this persistent term[c]. We fully agree with the absorptive parts of their integrals, but these discontinuities do not specify the subtraction constants—indeed none are needed, since all integrals are ultraviolet convergent. In that connection one can work out other amplitudes perturbatively (including pure odd-photon-number processes which do not vanish in this model) and easily find that no infinite renormalization constants are needed as $l \to 1$. Having considered all these methods, we have found that, aside from two minor oversights, all the methods in fact yield consistent results. This is reassuring since the integral considered is UV finite, so we would hope for agreement.

Having safely evaluated the vacuum polarization, we now wish to test the IR stability of the theory as we have done for SED and QED. To be able to do this we must first calculate the fermion self-energy. Despite the addition of the CS Lagrangian the theory still obeys the fermion Dyson-Schwinger equation given in equation (3.11), since the fermionic part of the Lagrangian remains unaltered, so

[b] Instead of setting $k^2 \to 0$ in their vacuum polarization, then letting $\lambda \to 0$, the correct behaviour is seen by performing the momentum integration, letting $\lambda \to 0$, and *then* taking asymptotic limits.

[c] This aspect of the analysis in Ref. [60], and its problems, have been alluded to in the non-abelian case by Pisarski and Rao [150].

the fermion self-energy may still be written as

$$\Sigma(p,\omega) = ie^2 \int d\!\!{}^-{}^3 k \; \gamma_\nu \frac{1}{\not p - \not k - \omega} \Gamma_\mu D^{\mu\nu}(k).$$

Since we want to remain within perturbation theory we will use the bare photon propagator and vertex, and since we are using dimensional regularization to evaluate this integral, we must use the $(2l+1)$-dimensional forms for both the photon propagator and vertex function, so that we must actually evaluate

$$\Sigma(p,\omega) = ie^2 \int d\!\!{}^-{}^{2l} k \; \gamma_{[\nu_1\cdots\nu_l]} \frac{1}{\not p - \not k - \omega} \gamma_{[\mu_1\cdots\mu_l]} D^{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k),$$

where $D^{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ is given in equation (4.10).

Since we find this integral finite when evaluated using dimensional regularization, and the limit $l \to 1$ may be safely taken, we obtain our (2+1)D results for the components of the self-energy discontinuity to be

$$\begin{aligned}\Sigma_{1I}(p^2,\omega^2) \;=\; & \frac{e^2}{8\pi(p^2)^{3/2}}\Bigg[\frac{\xi}{2}(\omega^2+p^2)\theta(p^2-\omega^2) - \left(\omega\mu + \frac{\mu^2}{2}\right)\theta(p^2-(\omega+\mu)^2) \\ & + \frac{1}{2\mu^2}(\omega^2-p^2)^2\left(\theta(p^2-(\omega+\mu)^2) - \theta(p^2-\omega^2)\right) \\ & + \frac{\omega}{\mu}(\omega^2-p^2)\left(\theta(p^2-(\omega+\mu)^2) - \theta(p^2-\omega^2)\right)\Bigg] \end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\Sigma_{2I}(p^2,\omega^2) \;=\; & \frac{-e^2}{8\pi\sqrt{p^2}}\Bigg[\xi\theta(p^2-\omega^2) + (2+\frac{\mu}{\omega})\theta(p^2-(\omega+\mu)^2) \\ & + \frac{(\omega^2-p^2)}{\omega\mu}\left(\theta(p^2-(\omega+\mu)^2) - \theta(p^2-\omega^2)\right)\Bigg]. \end{aligned} \tag{4.19}$$

We wish to ascertain whether the fermion self-energy discontinuity given by (4.18) and (4.19) will result in an equation for the fermion spectral function which is soluble within perturbation theory. Probably the easiest way to determine this is to consider the various terms above in turn to see how they affect the behaviour of the spinor GT equations, (3.16) and (3.17) which we need to solve. Obviously those terms in Σ_{1I} and Σ_{2I} above which contain at least one power of (ω^2-p^2) in the numerator will result in a zero contribution to $\Sigma_I(p^2,p^2)$, so they will not cause any IR problems. Similarly, if we combine those terms proportional to ξ via equation (3.15) we obtain

$$\Sigma_I^{\xi}(p,\omega) = \frac{e^2}{8\pi (p^2)^{3/2}} \frac{\xi}{2}(\omega^2 - p^2)\theta(p^2-\omega^2),$$

which also contains a factor of $(\omega^2 - p^2)$, and so will not lead to any IR problems.

Now we are only left with terms such as

$$\frac{e^2}{8\pi\sqrt{p^2}} \cdot 2\theta(p^2-(\omega+\mu)^2), \tag{4.20}$$

none of which has a factor of $(p^2-\omega^2)$, but all of which have a θ function with argument $p^2-(\omega+\mu)^2$ rather than $p^2-\omega^2$, and it is this argument which is important. Consider the effect of this term, if we take an integral such as (3.13) and substitute (4.20) for Σ_I, giving

$$(p-m)\rho^{(1)}(p) = \frac{e^2}{4\pi p}\int_m^\infty \frac{d\omega \rho^{(0)}(\omega)}{p-\omega}\theta(p^2-(\omega+\mu)^2). \tag{4.21}$$

Inserting $\rho^{(0)}(\omega) = \delta(\omega^2 - m^2)$ and evaluating the integral we obtain

$$(p-m)\rho^{(1)}(p) = \frac{e^2}{4\pi p}\frac{\theta(p^2-(m+\mu)^2)}{p-m}, \tag{4.22}$$

or

$$\rho^{(1)}(p) = \frac{e^2}{4\pi p}\frac{\theta(p^2-(m+\mu)^2)}{(p-m)^2}. \tag{4.23}$$

Inserting this again into (3.13) to go to the next order, we are confronted with

$$\begin{aligned}(p-m)\rho^{(2)}(p) &= \frac{e^4}{16\pi^2 p^2}\int \frac{d\omega\theta(p^2-(\omega+\mu)^2)}{(p-\omega)(\omega-m)^2\omega}\theta(\omega^2-(m+\mu)^2) \\ &= \frac{e^4}{16\pi^2 p^2}\int_{m+\mu}^{p-\mu}\frac{d\omega}{(p-\omega)(\omega-m)^2\omega}. \end{aligned} \tag{4.24}$$

Our integration region now avoids the areas of IR instability and it appears that we may safely evaluate this integral and obtain the behaviour of $\rho(\omega)$. If we combine (4.18) and (4.19) to give the full self-energy discontinuity, then insert it into (3.13), we are faced with

$$\begin{aligned}(\omega-m)\rho^{(i+1)}(\omega) = \frac{e^2}{8\pi\omega^2}\int \rho^{(i)}(\omega')d\omega' \Bigg[&\frac{\xi}{2}(\omega-\omega')\theta(\omega^2-\omega'^2) \\ &-\frac{(2\omega\omega'+\mu(\omega+\omega')+\mu^2/2)}{\omega-\omega'}\theta(\omega^2-(\omega'+\mu)^2) \\ &+\frac{(\omega^2-\omega'^2)}{2\mu^2}(\omega+\omega'+2\mu)[\theta(\omega^2-(\omega'+\mu)^2)-\theta(\omega^2-\omega'^2)]\Bigg].\end{aligned}$$

Taking $\rho^{(0)}(\omega') = \delta(\omega' - m)$ we find that

$$\begin{aligned}\rho^{(1)}(\omega) \;=\; & \frac{e^2}{8\pi\omega^2}\left[\frac{\xi}{2}\theta(\omega^2 - m^2) - \frac{(2\omega m + \mu(\omega + m) + \mu^2/2)}{(\omega - m)^2}\theta(\omega^2 - (\mu + m)^2)\right. \\ & \left. + \frac{(\omega^2 - m^2)}{2\mu^2}(\omega + m + 2\mu)[\theta(\omega^2 - (\mu + m)^2) - \theta(\omega^2 - m^2)]\right].\end{aligned}$$

and by inspection we can see that to the next order we will only encounter terms such as those discussed above, so the integral will indeed be finite. Furthermore, each successive iteration can be seen to introduce an extra shift by μ into the cut of $\rho^{(i)}$, so that we can predict that at higher orders, cuts in $\rho^{(i)}(\omega)$ will be at $\omega = m + i\mu$ and $\omega = m$.

Chapter 5

Dynamical Mass Generation in Odd Dimensions

In this chapter we will explore in more detail the idea of dynamical mass generation. This is the process where a particle develops a term proportional to some mass scale through quantum corrections to its propagator. We will deal first with the (2+1)-dimensional theory, then go on to consider theories in other odd dimensions [151].

5.1 Mass Generation in (2+1)D

We will begin our discussion of dynamical mass generation by considering the (2+1)-dimensional case, looking in detail at what happens when we allow the presence of particle masses into a non-parity-doubled theory, as intimated in section 4.1 above.

We first consider a theory containing a kinetic term and an interaction with massive fermions. This means we begin with the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\,\partial\!\!\!/ - m - e\gamma^{\mu}A_{\mu})\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{5.1}$$

and recall from Chapter 3 that this term of the form $m\bar{\psi}\psi$ will result in a violation of the parity invariance of the theory. We want to see the effect this parity-violating fermion mass will have on the photon, so we work out the photon vacuum polarization within perturbation theory,

$$\begin{aligned}\Pi_{\mu\nu}(k) &= ie^2 Z \int d\!\!{}^-{}^3p \,\mathrm{tr}[\gamma_\mu S(p)\Gamma_\nu(p,p-k)S(p-k)] \\ &\to ie^2 \int d\!\!{}^-{}^3p \,\frac{\mathrm{tr}[\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}-\not{k}+m)]}{(p^2-m^2)[(p-k)^2-m^2]}.\end{aligned} \tag{5.2}$$

We evaluate this by introducing Feynman parameters, then doing the momentum integration to give

$$\begin{aligned}\Pi_{\mu\nu}(k) &= (k_\mu k_\nu - k^2\eta_{\mu\nu})\frac{e^2}{2\pi}\int \frac{\alpha(1-\alpha)d\alpha}{\sqrt{m^2-k^2\alpha(1-\alpha)}} \\ &\quad + im\epsilon_{\lambda\mu\nu}k^\lambda \frac{e^2}{4\pi}\int \frac{d\alpha}{\sqrt{m^2-k^2\alpha(1-\alpha)}},\end{aligned} \tag{5.3}$$

or, once we evaluate the α-integration,

$$\begin{aligned}\Pi_{\mu\nu}(k) &= \frac{e^2}{16\pi}\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\left[\left(\sqrt{k^2}+\frac{4m^2}{\sqrt{k^2}}\right)\ln\left(\frac{2m+\sqrt{k^2}}{2m-\sqrt{k^2}}\right)-4m\right] \\ &\quad + im\epsilon_{\mu\nu\sigma}k^\sigma \frac{e^2}{4\pi\sqrt{k^2}}\ln\left(\frac{2m+\sqrt{k^2}}{2m-\sqrt{k^2}}\right).\end{aligned} \tag{5.4}$$

This result is identical to that of (4.15) above, which is not surprising since the interaction Lagrangian is the same in both cases, and the calculation of $\Pi_{\mu\nu}$ involves no photon propagators. The important thing to note is that $\Pi_{\mu\nu}$ has a term proportional to m, so we have effectively induced a photon mass through the presence of the fermion mass, as we expected to do [37, 45, 46]. This result is in sharp contrast to Chapter 3, where a mass was present, but parity was not violated due to the 4×4 γ matrices, which have

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\lambda] = 0,$$

whereas here we have

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\lambda] = \epsilon_{\mu\nu\lambda},$$

as we have explained in Appendix D. A similar straightforward calculation of the fermion self-energy graph produces the result

$$\Sigma(p) = \frac{e^2}{16\pi}\int \frac{dw}{w(\not{p}-w)}\left[\frac{\xi}{w}-\frac{4m}{(w-m)^2}\right]. \tag{5.5}$$

It should be noted that if we were to set $m = 0$, i.e. make the fermions massless, we would remove the parity violation, preventing the photon mass from being generated in (5.4). We can understand why this one loop correction to the photon mass (5.4) is all there is. If we consider a gauge transformation $\delta A \to \partial\chi$, our Lagrangian, ϵAF changes by a pure divergence, so the action remains invariant for all field configurations that vanish at ∞. If we then allow a fourth order interaction, which would correspond to a term like

$$\epsilon_{\mu\nu\lambda}A^{\mu}F^{\nu\lambda}F_{\alpha\beta}F^{\alpha\beta},$$

the invariance under the change of gauge will be broken, and similarly for higher-order interactions, so no other contributions are permitted. The absence of such corrections has been proven rigorously for the (2+1)-dimensional theory by Coleman and Hill [152]. The dynamical generation of this photon mass raises the question of whether the converse is true, that is, does the presence of a parity-violating photon mass term generate a fermion mass? There are two possible Lagrangians to consider; one with a Maxwell term together with the Chern-Simons term, the other the purely topological theory. We will consider each in turn.

First we consider a theory with Lagrangian

$$\mathcal{L} = \bar{\psi}(i\,\partial\!\!\!/ - e\gamma^{\mu}A_{\mu})\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^{\lambda} + \mathcal{L}_{GF}, \tag{5.6}$$

which is the same as in that in the previous chapter, but with zero bare fermion mass. Note that in (5.6) both the second and third terms are bilinear in the gauge field A, so the contribution from the Chern-Simons term will combine with the kinetic term in the bare photon propagator, giving

$$D_{\mu\nu}(k) = \frac{-1}{k^2-\mu^2}\left[\left(\eta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}\right) + \frac{i\,\mu}{k^2}\epsilon_{\mu\nu\lambda}k^{\lambda}\right] - \frac{\xi k_{\mu}k_{\nu}}{k^4}. \tag{5.7}$$

Using this and the bare fermion propagator,

$$S(p) = \frac{1}{p\!\!\!/},$$

we can now calculate the one-loop correction for the fermion, $\Sigma(p)$, giving,

$$\begin{aligned}
\Sigma(p) &= ie^2 \int d\!\!{}^-{}^3k \; \gamma_\nu \frac{1}{\not{p} - \not{k}} \gamma_\mu D^{\mu\nu}(k) \\
&= \frac{e^2 \not{p}}{16\pi p^2} \left[\frac{p^4 - \mu^4}{\mu^2 p} \ln\left(\frac{\mu + p}{\mu - p}\right) - \frac{2(p^2 - \mu^2)}{\mu} + \frac{\pi p^2 \sqrt{-p^2}}{\mu^2} - \frac{3}{2}\pi\xi\sqrt{-p^2} \right] \\
&\quad + \frac{e^2}{8\mu\pi} \left[\frac{p^2 - \mu^2}{p} \ln\left(\frac{\mu + p}{\mu - p}\right) - 2\mu + \pi\sqrt{-p^2} \right]. \qquad (5.8)
\end{aligned}$$

(This expression could have been directly evaluated by regarding ϵAF as an interaction, rather than combining it with the bare photon propagator as we have done.) It is the second term in (5.8) above, the one proportional to $e^2/8\mu\pi$, which behaves as the "mass" for the fermion.

We should notice that in the limit of small μ, the expression above reduces to

$$\Sigma(p) = \frac{e^2 \not{p}}{4\pi p^2}\left[\frac{\mu}{2} - \frac{3\pi\xi\sqrt{-p^2}}{8}\right] + \frac{e^2\mu}{8\sqrt{-p^2}},$$

and further, this expression will disappear when $\mu \to 0$ in the Landau gauge. To understand this result, it is easiest just to evaluate Σ for $\mu \to 0$ in the Landau gauge, which means taking

$$D_{\mu\nu}(k) = \frac{-1}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

and checking that

$$\begin{aligned}
\Sigma(p) &= ie^2 \int d\!\!{}^-{}^3k \; \gamma_\nu \frac{1}{\not{p} - \not{k}} \gamma_\mu D^{\mu\nu}(k) \\
&= ie^2 \int \frac{d\!\!{}^-{}^3k}{k^2(p+k)^2} \left[(2 - 2l) p \cdot k + \frac{p^2 \; p \cdot k}{k^2} \right] \\
&= 0.
\end{aligned}$$

Now we shall consider what happens when the initial Lagrangian has no gauge field kinetic energy but starts off life instead as a pure Chern-Simons Lagrangian interacting with massless fermions. In this case the Lagrangian takes the form

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - e\gamma^\mu A_\mu)\psi + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^\lambda + \mathcal{L}_{GF}, \qquad (5.9)$$

that is, the same as in (5.6) minus the second term. We want to quantize this theory using the canonical formalism [72,153,154]. We consider (5.9) as a Chern-Simons Lagrangian $\mathcal{L}_0$ plus fermion term $\mathcal{L}_\psi$ and gauge-fixing term $\mathcal{L}_{GF}$, then for simplicity consider only

$$\mathcal{L}_0 = \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^{\lambda}. \tag{5.10}$$

The field equation of (5.10) takes the form $\mu\epsilon_{\mu\nu\lambda}F^{\mu\nu} = 0$, which implies that $F^{\mu\nu}$ vanishes. Taking the 3D manifold M to be the direct product of a Riemann surface Σ and a real time axis, and selecting the Weyl gauge $A_0 = 0$ lets us re-express the Lagrangian in the simple form

$$\mathcal{L}_0 = \frac{\mu}{4}\epsilon_{ij}\dot{A}^i A^j, \tag{5.11}$$

which gives the Euler-Lagrange equations, $\dot{A}^i = 0$. The Gauss' law constraint is obtained from the variation of the action with respect to the zeroth component of the connection A^0, giving

$$\frac{1}{2}\epsilon^{ij}F_{ij} = \partial_1 A_2 - \partial_2 A_1 = 0.$$

Of course, if we included sources then the Gauss' law constraint would acquire an inhomogeneous term

$$\epsilon^{ij}\partial_i A_j = \rho/\mu,$$

which would lead to anyonic statistics [32, 33].

Equation (5.11) is first order in time derivatives, which leads to the non-vanishing quantum commutator

$$[A_i^a(x), A_j^b(y)] = \frac{i}{\mu}\epsilon_{ij}\delta^{ab}\delta(x-y). \tag{5.12}$$

Returning to (5.9), we find that in 2+1 dimensions, the pure CS term is this still bilinear in the gauge field, making it capable of launching a propagator,

$$D_{\mu\nu} = \frac{i\epsilon_{\mu\nu\lambda}k^{\lambda}}{\mu k^2} - \xi\frac{k_\mu k_\nu}{k^4}, \tag{5.13}$$

where we have taken account of gauge-fixing by introducing a parameter ξ. Evaluating the fermion self-energy now yields,

$$\Sigma(p) = e^2 \int \bar{d}^3 k \; \frac{\gamma_\mu(\not{p}-\not{k})\gamma_\nu}{(p-k)^2} D^{\mu\nu}(k) = -e^2\left[\frac{\xi\;\not{p}}{16\sqrt{-p^2}} + \frac{\sqrt{-p^2}}{8\mu}\right], \tag{5.14}$$

once again containing a term proportional to the mass scale $(e^2\sqrt{-p^2})/\mu$ where none previously existed. In the same vein we may compute the vacuum polarization correction to (5.13) and arrive at

$$\Pi_{\mu\nu}(k) = ie^2 \mathrm{tr} \int d\!\!\!^{-3}p \, \frac{\gamma_\mu \, \not{p}\gamma_\nu(\not{p}-\not{k})}{p^2(p-k)^2} = (-k^2\eta_{\mu\nu} + k_\mu k_\nu)\frac{e^2}{8\sqrt{-k^2}}, \tag{5.15}$$

which has the effect of leaving $D(k) \sim 1/k$. In higher orders of perturbation theory we may expect to find that

$$\Sigma(p) = \not{p}\, f(\frac{e^2}{\sqrt{-p^2}}, \frac{e^2}{\mu}) + \sqrt{-p^2}\; g(\frac{e^2}{\sqrt{-p^2}}, \frac{e^2}{\mu}),$$

and

$$\Pi_{\mu\nu} = (-k^2\eta_{\mu\nu} + k_\mu k_\nu)\; \pi(\frac{e^2}{\sqrt{-k^2}}, \frac{e^2}{\mu}),$$

where f, g and π are scalar functions of their arguments.

Both of the above results for Σ, (5.8) and (5.14), have been obtained by considering a single theory containing a conventional kinetic term weighted by a factor Z. The Maxwell + Chern-Simons theory would correspond to $Z = 1$, while the purely topological theory is obtained by taking the limit $Z \to 0$.

5.2 QED in Higher Odd Dimensions

The results of the previous section are not peculiar to 2+1 dimensions. In fact, in any odd dimension we might assume that we could induce a topological mass term. This section will generalize the results of the previous section to other odd dimensions. We begin as before with massive fermion QED, and by generalizing equations (1.1) and (1.2) to arbitrary D dimensions we obtain

$$\int d^Dx \Big[((\partial\phi)^2 - \mu^2\phi^2)/2 + \bar{\psi}(i\gamma.\partial - m)\psi - F^{\mu\nu}F_{\mu\nu}/4 + e\bar{\psi}\gamma.A\psi + (\partial_\mu\phi)^\dagger(ieA^\mu\phi)\Big], \tag{5.16}$$

and since we require a dimensionless action we find that the fields have mass dimension,

$$[\phi], [A] \sim M^{D/2-1}, \qquad [\psi] \sim M^{(D-1)/2},$$

leading to a coupling constant with dimensions $[e] \sim M^{2-D/2}$. We can see that if we consider odd D dimensions, the coupling e has an odd $\sqrt{M}$ scale, but this does not matter much for electrodynamics since we meet powers of e^2 in the perturbation expansion.

Generalising the discrete operations to arbitrary odd dimensions we find that a charge conjugation operator $\mathcal{C}$ with the property

$$\mathcal{C}\gamma_{[\mu_1\mu_2\cdots\mu_r]}\mathcal{C}^{-1} = (-1)^{[(r+1)/2]}(\gamma_{[\mu_1\mu_2\cdots\mu_r]})^T, \tag{5.17}$$

which always exists in even dimensions, cannot be defined at the odd values $D = (5, 9, 13, ...)$. As for parity $\mathcal{P}$, we can generalize (3.3) above to arbitrary odd dimensions, yielding

$$\begin{aligned}\mathcal{P}\psi(x_0, x_1, .., x_{D-2}, x_{D-1})\mathcal{P}^{-1} &= -i\eta\gamma_0\gamma_{D-1}\psi(x_0, -x_1, .., -x_{D-2}, x_{D-1}) \\ &= \eta\gamma_1\cdots\gamma_{D-2}\psi(x_0, -x_1, .., -x_{D-2}, x_{D-1}),\end{aligned} \tag{5.18}$$

where η is the intrinsic parity of the fermion field. Using this result one can easily check that a mass term like $m\bar{\psi}\psi$, which violated parity in (2+1)D, will continue to do so for arbitrary odd D. In arbitrary $2l+1$ dimensions we find that the topological term induced by this parity-violating fermion mass takes the form of an n-point function,

$$C\epsilon_{\mu_1\mu_2\cdots\mu_{2l+1}}A^{\mu_1}F^{\mu_2\mu_3}\cdots F^{\mu_{2l}\mu_{2l+1}}; \qquad n = l+1. \tag{5.19}$$

Note that this induced term conforms perfectly with the idea of charge conjugation above. When $2l+1 = 3$ and $\mathcal{C}$ is conserved, the topological term involves an even number $n = 2$ of photons. When $D = 5$ and $[e^2] \sim M^{-1}$, we encounter three photon lines but then $\mathcal{C}$ is no longer valid. When $D = 7$, $\mathcal{C}$-invariance becomes operative again and the number of photon lines is $n = 4$, and so on.

The result for the induced topological term in (2+1)D has already been quoted in Eq.(5.4). Looking at the next odd dimension, $D = 5$, the relevant one-loop graphs are shown in Figure 11, leading to the induced vertex,

$$\Gamma_{\lambda\mu\nu}(k, k') = -2ie^3\int \bar{d}^5p \frac{\mathrm{tr}[\gamma_\nu(\not{p}+m)\gamma_\mu(\not{p}+\not{k}+m)\gamma_\lambda(\not{p}-\not{k}'+m)]}{(p^2-m^2)((p+k)^2+m^2)((p-k')^2-m^2)}.$$

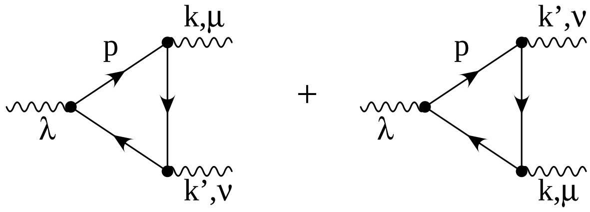


Figure 11: One-loop induction of a Chern-Simons amplitude in 5 dimensions.

Introducing Feynman parameters in the usual way to combine denominators and picking out the term with five gamma-matrices in the trace, we end up with

$$\begin{aligned}\Gamma_{\lambda\mu\nu}(k,k') &= -16ie^3m\int d\!\!{}^-{}^5p\frac{d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)\epsilon_{\lambda\mu\nu\rho\sigma}k^\rho k'^\sigma}{[p^2-m^2+k^2\alpha\beta+k'^2\gamma\alpha+(k+k')^2\beta\gamma]^3}\\ &= -\frac{e^3m\epsilon_{\lambda\mu\nu\rho\sigma}k^\rho k'^\sigma}{8\pi^2}\int_0^1\frac{d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)}{(m^2-k^2\alpha\beta-k'^2\gamma\alpha-(k+k')^2\beta\gamma)^{\frac{1}{2}}}.\end{aligned}\quad (5.20)$$

One can regard this amplitude as the five-dimensional description of the process $\pi^0\to 2\gamma$, because one of the indices (4) of the Levi-Civita tensor just corresponds to the standard pseudoscalar and the residual four indices (0 to 3) are the normal 4-vector ones. Just as with 2+1 QED, we see that the induced term in 4+1 QED vanishes with the fermion mass m.

We are now in a position to evaluate the topological vertex induced by the fermion mass term in arbitrary odd dimensions. This is given by considering the Feynman diagram shown in Figure 12, which represents the integral

$$\begin{aligned}\Gamma_{\lambda\mu_1\mu_3\cdots\mu_{2l+1}}(k_1,k_2,\cdots,k_n) &= -(n-1)!ie^n\int d\!\!{}^-{}^{2l+1}p\,\times\\ &\times\frac{\mathrm{tr}[(\not p+m)\gamma_{\mu_1}(\not p+\not k_1+m)\gamma_{\mu_3}\cdots(\not p+\sum_{i=1}^{n-1}\not k_i+m)\gamma_{\mu_{2l+1}}]}{(p^2-m^2)((p+k_1)^2+m^2)\cdots[(p+\sum_{i=1}^{n-1}k_i)^2-m^2]}.\end{aligned}\quad (5.21)$$

The only terms in (5.21) which will contribute to our induced mass term are those which contain $2l+1$ gamma matrices. We pick out all possible terms of this form from the numerator, then perform the trace operation on each. Combining the results, we find that all of the terms containing the internal momentum p in the numerator cancel, and we are left only with a term proportional to

$$m\epsilon_{\mu_1\mu_2\cdots\mu_{2l+1}}k_1^{\mu_2}k_2^{\mu_4}\ldots k_l^{\mu_{2l}}. \quad (5.22)$$

Since this is independent of p, we may ignore it while considering the momentum integration. We introduce $n = l+1$ Feynman parameters, $\alpha_1 \ldots \alpha_n$, one for each internal line in Figure 12, and the momentum integral becomes

$$\int d\!\!{}^-{}^{2l+1}p \frac{\left(\prod_{k=1}^{n} d\alpha_k\right) \delta(1-\alpha_1-\alpha_2\cdots-\alpha_n)\Gamma(n)}{\left(\sum_{i=1}^{n}\alpha_i[(p+\sum_{j=1}^{i-1}k_j)^2-m^2]\right)^n}. \tag{5.23}$$

If we let the denominator be equal to I^n, then we can factorize all the terms in I containing p, and using the fact that $\sum_{i=1}^n \alpha_i = 1$, we obtain

$$\begin{aligned} I &= p^2 - m^2 + 2\sum_{i=1}^{n}\alpha_i p\cdot\sum_{j=1}^{i-1}k_j + \sum_{i=1}^{n}\alpha_i\left(\sum_{j=1}^{i-1}k_j\right)^2 \\ &= \left(p+\sum_{i=1}^{n}\alpha_i\sum_{j=1}^{i-1}k_j\right)^2 - m^2 - \left(\sum_{i=1}^{n}\alpha_i\sum_{j=1}^{i-1}k_j\right)^2 + \sum_{i=1}^{n}\alpha_i\left(\sum_{j=1}^{i-1}k_j\right)^2. \end{aligned} \tag{5.24}$$

We now use the usual method associated with Feynman parametrization, that is, we make the shift $p \to p - \sum_{i=1}^n \alpha_i \sum_{j=1}^{i-1} k_j$. Then, by combining the last two terms in (5.24) and defining the quantity $k_{ab} \equiv \sum_{i=1}^{a-1} k_i - \sum_{i=1}^{b-1} k_i$, we can obtain the simple form

$$I = p^2 - m^2 + \sum_{i<j=1}^{n} k_{ij}^2 \alpha_i\alpha_j. \tag{5.25}$$

The quantity k_{ij} can be thought of as the momentum flow across the line which cuts the lines with parameters α_i, α_j. If we return to (5.23) and substitute (5.25) for I, we are left with an integral which has the form

$$\int \frac{d\!\!{}^-{}^{2l+1}p}{(p^2-M^2)^n},$$

where M is independent of p, so we can evaluate it, finally producing the result,

$$\begin{aligned} \Gamma_{\mu_1\mu_3\cdots\mu_{2l+1}}(k) &= -\frac{me^n i^{n-1}}{2(2\pi)^{n-1}}\epsilon_{\mu_1\mu_2\cdots\mu_{2l+1}}k_1^{\mu_2}k_2^{\mu_4}\ldots k_l^{\mu_{2l}} \times \\ &\quad \times \int_0^1 \Pi_{k=1}^n d\alpha_k \frac{\delta(1-\alpha_1\cdots-\alpha_n)}{\sqrt{m^2-\sum_{i<j=1}^n k_{ij}^2\alpha_i\alpha_j}}. \end{aligned} \tag{5.26}$$

One may readily check that this collapses to the results (5.3) and (5.20) for $D=3$ and $D=5$ respectively. It corresponds to the Chern-Simons term (5.19) where $C = e^n/2n!(4\pi)^{n-1}$ if one goes to the soft photon limit, always assuming $m \neq 0$.

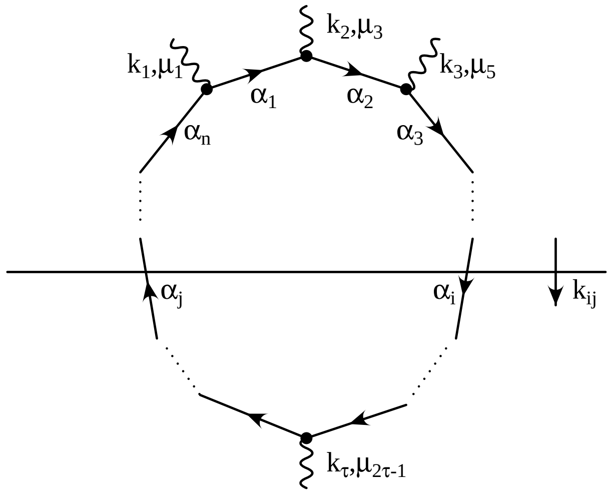


Figure 12: One-loop induction of a Chern-Simons term in $2l+1$ dimensions.

This result is still the only correction to the photon mass, since interactions like

$$\epsilon_{\mu_1\mu_2\cdots\mu_D} A^{\mu_1} F^{\mu_2\mu_3} \dots F^{\mu_{D-1}\mu_D} (F_{\rho\sigma}F^{\rho\sigma})^N; \qquad N \geq 1, \tag{5.27}$$

are forbidden by gauge-invariance and thus cannot be produced. Similarly, any two-loop contribution to the fundamental topological term would correspond to an integration of (5.27), with $N=1$, over one of the photon momenta. Since we have just concluded that (5.27) must be absent, we deduce that the induced topological term (5.19) cannot receive any two-loop (or higher-loop) quantum corrections. This means that our result gives the induced mass contribution exactly. This is similar to the Adler-Bardeen theorem for the axial anomaly, but in this case it is of little use as the theory becomes unrenormalizable for $D \geq 5$, due to the mass dimensions of e^2, except of course when the space-time is compact, e.g. in some Kaluza-Klein geometries.

Once again we will consider the possibility of the converse, that is, the *generation* of a fermion mass, through the presence of a photon mass, this time in higher odd dimensions. As in the previous section, we first consider a theory with both a Maxwell term and a Chern-Simons term, which has a Lagrangian of the form

$$\mathcal{L} = \bar{\psi}\gamma.(i\partial - eA)\psi - F_{\mu\nu}F^{\mu\nu}/4 + C\epsilon_{\mu_1\mu_2\cdots\mu_{2l+1}} A^{\mu_1} F^{\mu_2\mu_3} \cdots F^{\mu_{2l}\mu_{2l+1}}. \tag{5.28}$$

In such a theory we can be certain that the gauge field will propagate at the bare level in any number of dimensions, thanks to the Maxwell term. It is now necessary to regard the Chern-Simons contribution as an interaction, since for $l > 1$, it is no longer bilinear and cannot be incorporated into the bare propagator. Any fermion mass will appear in the calculation of the fermion self-energy, which to first order in C engenders a term of the form shown in Figure 13 below. If we

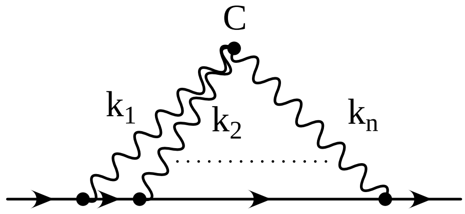


Figure 13: Induction of a fermion mass term through a topological interaction.

consider $D = 5$ to first order in $C\epsilon AFF$, we find

$$\begin{aligned}\Sigma(p) &= -ie^3C\int d\!\!{}^-{}^{2l+1}k\; d\!\!{}^-{}^{2l+1}k'\frac{\epsilon_{\mu\nu\lambda\alpha\beta}k^{\alpha}k'^{\beta}\gamma^{\mu}\gamma.(p-k-k')\gamma^{\nu}\gamma.(p-k)\gamma^{\lambda}}{k^2k'^2(k+k')^2(p-k-k')^2(p-k)^2}\\ &= \frac{3\Gamma(2-2l)p^4e^3C}{(16)^3\pi^4},\end{aligned} \tag{5.29}$$

which contains a non-zero mass term and no kinetic term. Notice that if in (5.29) we let $C \to 0$, removing the parity-violating photon mass, the fermion mass will also disappear. Unfortunately this five dimensional self-energy is divergent as $l \to 2$, which is not too surprising since $e^2 \sim M^{-1}$, and the theory is unrenormalizable. There is likewise a 2-loop contribution of the same type to the photon self-energy, but this cannot add a parity-violating part to Π because such a term would violate gauge-invariance for $D = 5$ as we have already explained above.

Finally we turn to the purely topological theory in $2l + 1$ dimensions, where the Lagrangian takes the form

$$\mathcal{L} = \bar{\psi}\gamma.(i\partial - eA)\psi + C\epsilon_{\mu_1\mu_2\cdots\mu_{2l+1}}A^{\mu_1}F^{\mu_2\mu_3}\cdots F^{\mu_{2l}\mu_{2l+1}}.$$

Such theories have been considered [155–159], mainly in regard to their quantization and connections to conformal field theory. We wish instead to continue to pursue our perturbative investigation, since if $l \geq 2$, that is, if the theory is in five or more dimensions, then the situation becomes radically different from the previous theory (5.28), since the Chern-Simons term is no longer bilinear in the gauge field and alone cannot give rise to a propagator for the photon. Instead we must resort to quantum corrections to generate photon propagation. If we consider the (4+1)-dimensional theory to first order we may obtain a contribution from the vacuum polarization graph (with massless fermions), namely

$$\Pi_{\mu\nu}(k) = -ie^2 \int d\!\!{}^-{}^5 p \gamma_\mu \frac{1}{\not p} \gamma_\nu \frac{1}{\not p + \not k}.$$

This produces a hard quantum loop contribution

$$D_{\mu\nu}(k) = (-\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4})\frac{512\pi}{3e^2\sqrt{-k^2}} - \xi \frac{k_\mu k_\nu}{k^4}. \tag{5.30}$$

Taken with the trilinear Chern-Simons interaction this can produce a further vacuum polarization effect from the gauge field itself, namely

$$\Pi_{\mu\nu}(k) = i\left(\frac{512\pi C}{3e^2}\right)^2 \int \frac{d\!\!{}^-{}^5 k'}{(k'^2(k-k')^2)^{3/2}} \epsilon_{\mu\rho\sigma\alpha\beta} k^\alpha k'^\beta \epsilon_\nu{}^{\rho\sigma\gamma\delta} k_\gamma k'_\delta,$$

which corresponds to the Feynman diagram shown in Figure 14. To evaluate this,

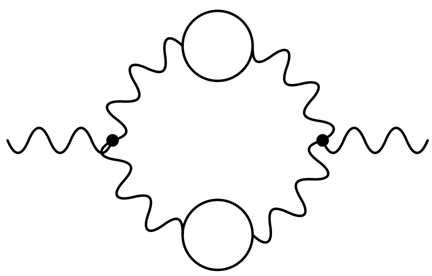


Figure 14: Gauge field contribution to vacuum polarization.

we first take advantage of gauge-invariance to extract a factor $(-\eta_{\mu\nu} + k_\mu k_\nu / k^2)$, which we are always permitted to do. Then, by contracting

$$\epsilon_{\tau\rho\sigma\alpha\beta}\epsilon^{\tau\rho\sigma\gamma\delta} = \eta_\alpha^\gamma \eta_\beta^\delta - \eta_\beta^\gamma \eta_\alpha^\delta$$

we are left with an integral of the form

$$\Pi_{\mu\nu} = i\left(\frac{512\pi C}{3e^2}\right)^2 (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \int \frac{d\!\!{}^-{}^5 k'}{(k'^2(k-k')^2)^{3/2}} \left(k^2 k'^2 - (k\cdot k')^2\right). \tag{5.31}$$

Since there are no particle masses in this expression, it is straightforward to evaluate (5.31) by the usual method, i.e. introducing Feynman parameters and evaluating first the momentum integral, then the (in this case trivial) parametric integral. Doing this yields

$$\Pi_{\mu\nu} = (\eta_{\mu\nu}k^2 - k_\mu k_\nu)\left(\frac{512C}{3e^2}\right)^2 \frac{\sqrt{-k^2}}{12}.$$

Interestingly though, (5.30) will no longer give birth to a mass-like fermion self-energy at one loop level, since five gamma matrices are now needed to obtain that. This means we have to consider two loop effects, either to order e^4 or to first order in the Chern-Simons coupling C, as sketched in Figure 13. Quite generally we may anticipate in 4+1 dimensions that the fermion and photon will behave as

$$\Sigma(p) = \gamma.p\ f(e^2\sqrt{-p^2}, C/e^3) + \sqrt{-p^2}\ g(e^2\sqrt{-p^2}, C/e^3)$$

and

$$\Pi_{\mu\nu} = (-k^2\eta_{\mu\nu} + k_\mu k_\nu)\ \pi(e^2\sqrt{-k^2}, C/e^3).$$

However, we must remember that higher-order contributions in e^2 and C are very likely unrenormalizable again here and so probably of academic interest. Still, our discussion does indicate the nature of the parity-violating contributions in these models and how they arise from a single source.

Chapter 6

Conclusion

6.1 Summary

In this thesis we have endeavoured to examine gauge theory in three dimensions. We have used both perturbation theory and non-perturbative techniques, and witnessed some of the unusual behaviour one encounters in such theories.

Chapter 2 served as an introduction to some of the problems of three dimensional theories. The IR catastrophe was seen to prevent the perturbation expansion from providing a finite result, so we were forced to dress the photon propagator with massless fermions within the gauge technique to allow calculations. We found that using this non-perturbative approach, we were able to obtain a IR finite spectral function for the meson, which turned out to be gauge invariant. In order to understand this we derived the gauge covariance relation between mesons in two covariant gauges, and found that this equation agreed with our result.

The spinor version of QED was considered in Chapter 3, and the IR behaviour of the perturbative approach investigated again. As in the scalar case, it was found that using the perturbation expansion led irrevocably to the IR catastrophe, so the photon was dressed, and the fermion behaviour probed through the GT. The gauge covariance relations were obtained once again, and the expressions for the fermion spectral function obeyed these relations in both the perturbative and non-perturbative case.

In Chapter 4, we relaxed the notion of parity conservation in order to permit the presence of a Chern-Simons term in the Lagrangian. We found that due to the presence of the Levi-Civita tensor, $\epsilon_{\mu\nu\lambda}$, which is inherently 3-dimensional, the analytic continuation required in dimensional regularization became difficult. To solve this we recast the Lagrangian into a form which exists in arbitrary $2l + 1$ dimensions, permitting the theory to be regularized, and perturbation calculations to be performed. Chern-Simons theory was found to be different from those considered earlier in that the perturbative approach proved successful, and so we obtained an IR stable result.

Dynamical mass generation was considered in Chapter 5. We found that if we allowed a parity-violating fermion mass to be present, a corresponding photon mass was generated through quantum corrections. Conversely, if a topological photon mass term was present, either on its own or together with a kinetic term, it engendered a mass in the fermion, through dynamical effects. We then went on to consider a generalization of these results to arbitrary odd dimensions. We found that once again the presence of a fermion led to a topological mass term, and that in a theory with a topological photon mass term plus a kinetic term, we generated a fermion mass term. The interesting thing was that in the purely topological theory in greater than three dimensions, no fermion mass was generated at the one loop order, due to the absence of a bare photon propagator.

6.2 Outlook

This thesis has looked at various different abelian theories in 2+1 (and higher) dimensions, but the fact that it was necessary to use the word abelian raises an obvious question. What about non-abelian theories? This question leads to several unsolved problems since calculations in non-abelian theories are more complicated, due mainly to the presence of the ghost sector.

If we consider extending the present work to its non-abelian counterpart and performing similar calculations, we would need to find a non-abelian form of the GT. This is difficult since in an abelian theory it is the simple form of the WGT identity that allows the *ansatz* for the vertex to take such an elegant form. In the non-abelian theory, the generalized WGT or Slavnov-Taylor identities [116, 117] involve ghost contributions which complicate their form, no longer permitting the construction of an *ansatz* such as (2.13). Several researchers of the GT in QCD [160–163] have avoided the use of the spectral representation altogether, constructing their *ansatz* for the three-gluon vertex through various different means, but their results have been contradictory. Cornwall offered an alternative in the pinch technique [38, 39] which is in a sense more satisfactory, since he re-arranged diagrams contributing to physical (gauge-invariant) processes in such a way that corrections to the effective gluon propagator became gauge-invariant, making effective quantities obey abelian-like WGT identities. This procedure would facilitate the use of the spectral form of the GT.

Assuming for the moment that we could satisfactorily set up a consistent structure for studying the non-abelian generalizations of our theories, we are still faced with an obstacle. If we were to study a non-abelian CS theory, and wanted as in Chapter 4 to use dimensional regularization, the formulation of the theory, now with a Lagrangian of the form

$$\epsilon_{\mu\nu\lambda}(A^\mu\partial^\nu A^\lambda + \frac{2}{3}A^\mu A^\nu A^\lambda),$$

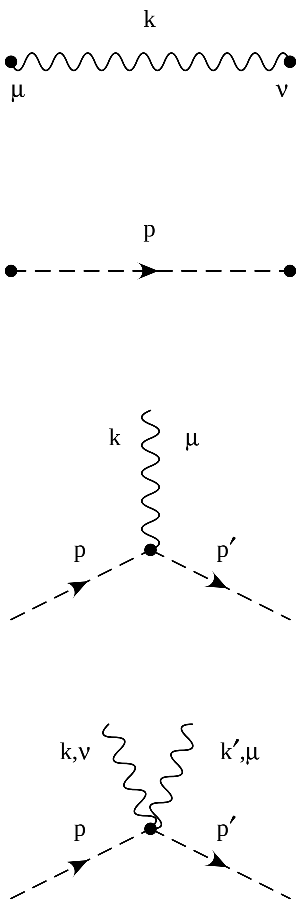
becomes extremely difficult. The solution we put forward for the abelian theory is not readily generalized since if we let the gauge field develop l indices and the ϵ tensor become $2l+1$ dimensional, the second term in the above expression will not be consistent away from $l = 1$, so another approach is required. Our approach to CS theory could however be extended to so-called BF theories [164, 165], where only the auxiliary B field would be extended (although we would still need to consider an l-fold nest of ghost-for-ghost mechanisms). This may also be a useful test of the logic of extending the labels of one or more of the fields.

Appendix A: Feynman Rules

In this appendix, Feynman rules are given for some of the theories considered in this thesis.

Scalar Electrodynamics

First we give the Feynman rules for the scalar version of quantum electrodynamics, shown below.



$$: \quad -i\left(\frac{\eta^{\mu\nu}}{k^2}-(1-\xi)\frac{k^\mu k^\nu}{k^4}\right)$$

$$: \quad \frac{i}{p^2-m^2+i\epsilon}$$

$$: \quad -i\,e(p^\mu+p'^\mu)$$

$$: \quad 2i\,e^2\eta^{\mu\nu}$$

Figure 15: Feynman rules for SED.

They are obtained from the Lagrangian

$$\mathcal{L}=[(\partial_\mu+ieA_\mu)\phi]^\dagger[(\partial^\mu+ieA^\mu)\phi]-m^2\phi^\dagger\phi-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{4\xi}(\partial_\mu A^\mu)^2,$$

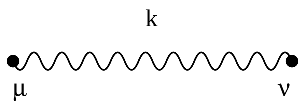
by taking suitable functional derivatives with respect to the fields. Only internal lines are included, and all internal momenta must be integrated over.

Spinor Electrodynamics

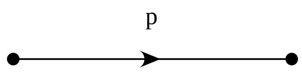
Here we give the Feynman rules for the spinor version of quantum electrodynamics, which are obtained from the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\,\partial\!\!\!/ - m - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4\xi}(\partial_\mu A^\mu)^2,$$

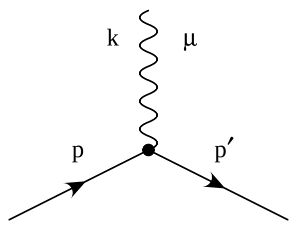
and given in Figure 16 below.



$$: \qquad -i\left(\frac{\eta^{\mu\nu}}{k^2} - (1-\xi)\frac{k^\mu k^\nu}{k^4}\right)$$



$$: \qquad \frac{i}{p\!\!\!/ - m + i\epsilon}$$



$$: \qquad -i\, e\gamma_\mu$$

Figure 16: Feynman rules for QED.

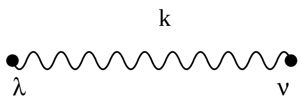
Once again, all internal momenta must be integrated over, and also a factor of -1 must be added for each fermion loop.

Chern-Simons Theory

In this theory, we use the Lagrangian

$$\mathcal{L}_{(2+1)\mathrm{D}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^{\lambda} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\,\partial\!\!\!/ - m - e\gamma^\mu A_\mu)\psi$$

to obtain our Feynman rules, and the rules for the fermion and for the vertex are equivalent to those given for QED above. Only the photon changes, becoming as shown in Figure 17.



$$: \quad \frac{-i}{k^2-\mu^2}\left[(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \frac{i\mu}{k^2}\epsilon^{\lambda\nu\alpha}k_\alpha\right] - \frac{i\xi k^\lambda k^\nu}{k^4}$$

Figure 17: Feynman rule for the Chern-Simons photon.

Appendix B: Dimensional Regularization

The purpose of this appendix is to illustrate the steps involved in using dimensional regularization [135–139] to evaluate apparently UV divergent momentum integrals. To do so, we will first take an explicit example, equation (2.27) from section 2.2, namely

$$\Pi_{\mu\nu}(k) = -ie^2 \int đ^3 p \; \left[\frac{(2p-k)_\mu (2p-k)_\nu - 2\eta_{\mu\nu}[(p-k)^2 - m^2]}{(p^2-m^2)[(p-k)^2-m^2]} \right],$$

and outline the steps in obtaining its solution, following the technique as it is outlined in [139]. The first step is to rewrite the integral in an arbitrary $2l$ dimensional form. In this case it is straightforward, but the discussion of a more difficult case is given in Chapter 4. The integral becomes

$$\Pi_{\mu\nu}(k) = -ie^2 \int đ^{2l} p \; \left[\frac{(2p-k)_\mu (2p-k)_\nu - 2\eta_{\mu\nu}[(p-k)^2 - m^2]}{(p^2-m^2)[(p-k)^2-m^2]} \right]. \tag{B.1}$$

The gauge invariance of the vacuum polarization ensures that we can rewrite this integral as the product of a scalar integral and a tensor, namely

$$\Pi_{\mu\nu}(k) = (-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})\Pi(k^2), \tag{B.2}$$

where

$$\begin{aligned} \Pi(k^2) &= -\frac{1}{(2l-1)} \Pi_\mu^{\ \mu}(k) \\ &= \frac{ie^2}{2l-1} \int đ^{2l} p \; \left[\frac{(2p-k)\cdot(2p-k)}{(p^2-m^2)[(p-k)^2-m^2]} - \frac{2l}{(p^2-m^2)} \right]. \end{aligned} \tag{B.3}$$

The first term above is still too complicated. We need to write the denominator in a different form, which we do by utilizing the identity [166]

$$\prod_{r=1}^{N} \frac{\Gamma(\nu_r)}{A_r^{\nu_r}} = \left(\prod_r \int_0^1 \alpha_r^{\nu_r - 1} d\alpha_r \right) \frac{\delta(1-\alpha_1 - \cdots - \alpha_N)\Gamma(\nu_1 + \cdots + \nu_N)}{[A_1\alpha_1 + \cdots + A_N\alpha_N]^{\nu_1 + \cdots + \nu_N}}, \tag{B.4}$$

which introduces the Feynman parameters, α_r. In equation (B.3) above, we need only the lowest in this series of equations, which takes the visually less disturbing form,

$$\frac{1}{AB} = \int_0^1 \frac{d\alpha}{[A\alpha + B(1-\alpha)]^2}, \tag{B.5}$$

which in terms of the denominators in the first term in (B.3) becomes, after factoring terms containing p,

$$\frac{1}{(p^2-m^2)[(p-k)^2-m^2]} = \int_0^1 \frac{d\alpha}{[(p-k\alpha)^2+k^2\alpha(1-\alpha)-m^2]^2}, \tag{B.6}$$

so that (B.3) becomes

$$\begin{aligned}\Pi(k^2) &= \frac{ie^2}{2l-1}\int \frac{đ^{2l}p\, d\alpha\,(2p-k)\cdot(2p-k)}{[(p-k\alpha)^2+k^2\alpha(1-\alpha)-m^2]^2} \\ &\quad -\frac{2lie^2}{2l-1}\int \frac{đ^{2l}p}{(p^2-m^2)}.\end{aligned} \tag{B.7}$$

Since we are integrating over all values of the internal momentum p, from $-\infty \to +\infty$, we may perform the shift (in the numerator as well) $p \to p + k\alpha$. The denominator is now of the form $(p^2-M^2)^2$, where M is a notional "mass" containing no p dependence, and this permits us now to discard terms which are odd in p, since they will cancel in the two halves of the integration region. Terms which are even in p are simplified according to

$$\int \frac{p_\mu p_\nu đ^{2l}p}{f(p^2)} = \frac{\eta_{\mu\nu}}{2l}\int \frac{p^2 đ^{2l}p}{f(p^2)},$$

(and also using

$$\int \frac{p_\mu p_\nu p_\kappa p_\lambda đ^{2l}p}{f(p^2)} = \frac{\eta_{\mu\nu}\eta_{\kappa\lambda}+\eta_{\mu\kappa}\eta_{\nu\lambda}+\eta_{\mu\lambda}\eta_{\kappa\nu}}{2l(2l+2)}\int \frac{p^4 đ^{2l}p}{f(p^2)}$$

when terms containing four indices occur) so that only terms of the form

$$\int đ^{2l}p \frac{(p^2)^T}{[p^2-M^2]^\Sigma}$$

remain. In our explicit example, (B.7) becomes

$$\begin{aligned}\Pi(k^2) &= \frac{ie^2}{2l-1}\int \frac{đ^{2l}p\, d\alpha\,(4p^2+k^2(1-2\alpha)^2)}{[p^2+k^2\alpha(1-\alpha)-m^2]^2} \\ &\quad -\frac{2lie^2}{2l-1}\int \frac{đ^{2l}p}{(p^2-m^2)}.\end{aligned} \tag{B.8}$$

This momentum integral can now be evaluated [139] via

$$-i\int đ^{2l}p \frac{(p^2)^T}{[p^2-M^2]^\Sigma} = \frac{(-1)^{T-\Sigma}\Gamma(l+T)\Gamma(\Sigma-T-l)}{(4\pi)^l\,\Gamma(l)\Gamma(\Sigma)(M^2)^{\Sigma-T-l}}, \tag{B.9}$$

which results in

$$\begin{aligned}\Pi(k^2) &= \frac{-e^2}{2l-1}\int\frac{d\alpha}{(4\pi)^l}\left[\frac{-4l\Gamma(1-l)}{[m^2-k^2\alpha(1-\alpha)]^{1-l}}+\frac{k^2(1-2\alpha)^2\Gamma(2-l)}{[m^2-k^2\alpha(1-\alpha)]^{2-l}}\right]\\ &\quad -\frac{2le^2\Gamma(1-l)}{(2l-1)(4\pi)^l(m^2)^{1-l}}\ . \end{aligned} \tag{B.10}$$

Having safely evaluated the momentum integrals, we may now take the limit $l \to 3/2$, and obtain the result for Π, yielding

$$\Pi_{\mu\nu}(k) = -\frac{e^2}{16\pi}(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{k^2})\left[4m+\left(\sqrt{k^2}-\frac{4m^2}{\sqrt{k^2}}\right)\ln\left(\frac{2m+\sqrt{k^2}}{2m-\sqrt{k^2}}\right)\right]\ , \tag{B.11}$$

which is just (2.28) above.

Appendix C: A Dispersive Calculation

It is instructive to work out the discontinuities of some of the Feynman integrals in QED in arbitrary $2l$ dimensions using a dispersion relation approach in order to understand what is peculiar to $2l = 3$, as well as seeing an alternative to the method explained in the previous appendix. We interpret the rules of dimensional continuation in the usual way, taking

$$i\Delta_c(x) = \left(\frac{m}{r}\right)^{l-1} \frac{K_{l-1}(mr)}{(2\pi)^l}\ ; \quad r^2 = -x^2 \tag{C.1}$$

and

$$\Delta_c(p) = \frac{1}{p^2 - m^2}\ . \tag{C.2}$$

The general phase space integral is the p^2 discontinuity of the integral

$$F(p) = -i \int đ^{2l}k\ \frac{1}{(k^2-\mu^2)[(k-p)^2 - m^2]}, \tag{C.3}$$

that is

$$\frac{1}{\pi}\Im F(p) = 2\pi \int đ^{2l}k\ \delta_+[(k-p)^2 - m^2]\ \delta_+[k^2-\mu^2]. \tag{C.4}$$

Now we make the hyperbolic polar decomposition (analogous to ordinary spherical polars)

$$d^{2l}k \to k^{2l-1}dk d\theta_{2l-1} \sinh\theta_{2l-2} d\theta_{2l-2} \cdots \sinh^{2l-2}\theta_1 d\theta_1,$$

transforming our integral into

$$\int \frac{k^{2l-1}dk\ d\theta_{2l-1} \sinh\theta_{2l-2} d\theta_{2l-2} \cdots \sinh^{2l-2}\theta_1 d\theta_1}{(2\pi)^{2l-1}.2\mu} \delta_+[(k-p)^2 - m^2]\ \delta_+[k^2-\mu^2],$$

where $\theta_1 \cdots \theta_{2l-2}$ run from 0 to π and θ_{2l-1} runs from 0 to 2π. We use the $\delta_+[k^2 - \mu^2]$ to evaluate the k integration, choose p such that $k \cdot p = \sqrt{k^2}\sqrt{p^2}\cosh\theta_1$, and exploit the fact that $\int_0^\pi sinh^{2\mu}\theta\, d\theta = B(1/2, \mu + 1/2)$ to evaluate all but the last angular integration, yielding

$$\begin{aligned}\frac{1}{\pi}\Im F(p) &= \frac{\mu^{2l-2}\pi^{(2l-3)/2}}{2.(2\pi)^{2l-2}\Gamma(\frac{n-1}{2})} \int_0^\pi \sinh^{2l-2}\theta_1 d\theta_1 \delta_+[\mu^2 + 2\mu\sqrt{p^2}\cosh\theta_1 + p^2 - m^2] \\ &= \frac{(p^2)^{1-l}\Delta^{2l-3}}{(16\pi)^{l-\frac{1}{2}}\Gamma(l-\frac{1}{2})}\theta(p^2 - (m+\mu)^2), \end{aligned}\tag{C.5}$$

where $\Delta^2 \equiv p^4 + m^4 + \mu^4 - 2m^2\mu^2 - 2\mu^2p^2 - 2m^2p^2$ is the usual triangle function. Also it should be noted that the charge coupling e has dimensions M^{2-l}, given the usual dimensionality of meson and fermion fields. Straightforward calculations use (C.5) to yield the following results. Defining $\Pi_{\mu\nu}(k) \equiv (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\pi(k^2)$, the polarization function Π receives the contributions

$$\begin{aligned} \Pi_s(k^2) &= \frac{e^2\Gamma(2-l)k^2(m^2)^{\ell-2}}{3(4\pi)^l}\ {}_2F_1(2-l,1;\frac{5}{2};\frac{k^2}{4m^2}), \\ \Pi_f(k^2) &= \frac{e^2 2^{[l+a]}k^2(m^2)^{l-2}}{3(4\pi)^l}\ {}_2F_1(2-l,2;\frac{5}{2};\frac{k^2}{4m^2}), \end{aligned} \tag{C.6}$$

from the charged scalar meson (mass m) and the (doubled) charged fermion (mass m) respectively, where $a = 1$ for odd dimensions, $a = 0$ for even dimensions. The absorptive parts of these contributions are

$$\begin{aligned} \frac{1}{\pi}\Im\Pi_s(k^2) &= \frac{e^2(k^2-4m^2)^{l-\frac{1}{2}}}{(16\pi)^{l-\frac{1}{2}}\sqrt{k^2}\Gamma(l+\frac{1}{2})}\theta(k^2-4m^2) \\ \frac{1}{\pi}\Im\Pi_f(k^2) &= \frac{-e^2 2^{[l+a]}(k^2-4m^2)^{l-\frac{3}{2}}}{(16\pi)^{l-\frac{1}{2}}\sqrt{k^2}\Gamma(l+\frac{1}{2})}\{2m^2+(l-1)k^2\}\theta(k^2-4m^2)\ . \end{aligned} \tag{C.7}$$

The formulae (C.6), (C.7) are of course consistent with one another bearing in mind that

$$\frac{1}{\pi}\Im F(a,b;c;z) = \frac{\Gamma(c)\theta(z-1)(z-1)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(1-a-b+c)}F(c-a,c-b;c-a-b+1;1-z)$$

$$\begin{aligned} F(a,b;a;z) &= (1-z)^{-b} \\ F(a,b;a-1;z) &= \left[1-\left(1-\frac{b}{a-1}\right)z\right](1-z)^{-1-b}\ . \end{aligned}$$

One may similarly evaluate the complete expressions for the meson and fermion self-energies due to (massless) photon emission and reabsorption. We shall not quote the complete result but only the discontinuities of those expressions which are of direct interest. To order e^2 the meson self-energy has the discontinuity,

$$\frac{1}{\pi}\Im\Sigma(p) = \frac{2e^2(p^2)^{1-l}(p^2-m^2)^{2l-3}(p^2+m^2)[1-2l+\xi(2l-3)]}{(16\pi)^{l-\frac{1}{2}}\Gamma(l-\frac{1}{2})}\theta(p^2-m^2) \tag{C.8}$$

and the fermion self-energy has the discontinuity,

$$\frac{1}{\pi}\Im\Sigma(p) = \frac{e^2(p^2-m^2)^{2l-3}}{(16\pi)^{l-\frac{1}{2}}\Gamma(l-\frac{1}{2})(p^2)^{l-1}}\left[\frac{\xi(p^2-m^2)\,\not{p}}{p^2}+2-2\xi-4l\right]\theta(p^2-m^2). \tag{C.9}$$

As mentioned in Chapter 2, we can see why the meson self energy discontinuity is gauge invariant to this order, since in (C.8), the term containing ξ happens to carry a factor of $2l-3$.

Appendix D: Dirac Gamma Matrices in Odd Dimensions

In order to evaluate integrals involving Dirac γ-matrices we require knowledge of their behaviour through certain identities, which we will outline below.

In any dimension D, the algebra of Dirac γ-matrices is generated by

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{D.1}$$

where μ and ν run from $0, \cdots, D-1$ and our metric is defined by

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \cdots).$$

The identity (D.1) above permits us to write down some properties of γ-matrices in any dimension.

First, we can write the contractions of γ's in D dimensions,

$$\begin{aligned}
\gamma_\mu\gamma^\mu &= D \\
\gamma_\mu\gamma_\nu\gamma^\mu &= (2-D)\gamma_\nu \\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu &= 2\gamma_\beta\gamma_\alpha + (D-2)\gamma_\alpha\gamma_\beta \\
&\equiv 4\eta_{\alpha\beta} + (D-4)\gamma_\alpha\gamma_\beta \\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\lambda\gamma^\mu &= (4-D)\gamma_\alpha\gamma_\beta\gamma_\lambda - 2\gamma_\lambda\gamma_\beta\gamma_\alpha, \\
&\text{etc.}
\end{aligned} \tag{D.2}$$

It is also possible in arbitrary D dimensions to write the trace of an even number of γ's as

$$\begin{aligned}
\frac{1}{2^{D/2}}\mathrm{tr}[\gamma_\mu\gamma_\nu] &= \eta_{\mu\nu} \\
\frac{1}{2^{D/2}}\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta] &= \eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\alpha\nu}.
\end{aligned} \tag{D.3}$$

More complicated trace expressions are sometimes required, which we obtain from Akyeampong and Delbourgo [148]. In order to explain them we define the antisymmetric product,

$$\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r]} \equiv \sum_{\mu_r\, perm} (-1)^p \gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r},$$

for which

$$\mathrm{tr}[\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r]}(\gamma)^s] = 0 \text{ unless } s = r.$$

This definition lets us express the results,

$$\frac{1}{2^{D/2}}\mathrm{tr}[\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r]}\gamma^{\nu_1}\gamma^{\nu_2}\cdots\gamma^{\nu_r}] = (-1)^{[r/2]}\delta^{[\nu_1}_{[\mu_1}\cdots\delta^{\nu_r]}_{\mu_r]}$$

and

$$\frac{1}{2^{D/2}}\mathrm{tr}[\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r]}\gamma_\alpha\gamma^{\nu_1}\gamma^{\nu_2}\cdots\gamma^{\nu_r}\gamma_\beta] =$$
$$= (-1)^{[3r/2]}\Big[\eta_{\alpha\beta}\delta^{[\nu_1}_{[\mu_1}\cdots\delta^{\nu_r]}_{\mu_r]} - \delta^{[\nu_1}_{\alpha}\eta_{\beta[\mu_1}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_r]}_{\mu_r]} - \delta^{[\nu_1}_{\beta}\eta_{\alpha[\mu_1}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_r]}_{\mu_r]}\Big].$$

Now we wish to consider properties dependent on the "oddness" of the dimensionality. When D is even it is well-known that the 'γ' have size $2^{[D/2]} \times 2^{[D/2]}$ and there exists a 'γ_5' matrix which is the product of all the different D γ's and which anticommutes with each γ_μ; one can always arrange it to have square -1, like all the space-like γ. It is not so well-known that in one higher dimension, there are two choices when realizing the algebra (D.1). The first choice is to let the size of the γ's remain the same, merely letting the 'γ_5' matrix become the last element of the D Dirac matrices. For instance, in three dimensions one can take the two-dimensional Pauli matrices $\gamma_0 = \sigma_3, \gamma_1 = i\sigma_1$ and simply append $\gamma_2 =$ 'γ_5' $= i\gamma_0\gamma_1 = -i\sigma_2$ to complete the set, without altering the size of the representation. Similarly, in five dimensions one can take the usual four-dimensional ones and just append $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ as the fifth component.

It is important to notice that it is possible to get a non-zero trace from the product of an odd number of gamma-matrices, if there are exactly D 'γ's, since at least

$$\mathrm{tr}[\gamma_{\mu_1}\gamma_{\mu_2}...\gamma_{\mu_D}] = (2i)^{(D-1)/2}\epsilon_{\mu_1\mu_2\cdots\mu_D}.$$

It is exactly this property which permits the generation of a Chern-Simons term in 3 dimensions. Another property worth remembering in odd dimensions is that if one constructs suitably normalized antisymmetric products of r matrices, $\gamma_{[\mu_1\mu_2\cdots\mu_r]}$ (the total set of these from $r = 0$ to $r = D$ generates a complete set into which any $2^{[D/2]} \times 2^{[D/2]}$ matrix can be expanded) then there exists the relation,

$$\gamma_{[\mu_1\mu_2\cdots\mu_r]} = \frac{i^{(D-1)/2}}{(D-r)!}\epsilon_{\mu_1\mu_2\cdots\mu_D}\gamma^{[\mu_{r+1}\mu_{r+2}\cdots\mu_D]}.$$

This often helps in simplifying products of matrices.

The other option is to use the so-called parity-doubled form of the Dirac matrices. This was discussed for the 3 dimensional case in Chapter 3, and can easily be generalized to the arbitrary odd dimensional case. Basically, it involves embedding the γ matrices of the first approach within a $2^D \times 2^D$ basis, so that

$$\Gamma^\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix},$$

where μ still runs from $0 \cdots D-1$. In this basis, the parity invariance of particle masses has been restored, preventing Chern-Simons terms being generated. In this case, the trace of *any* odd number of Γ's is equal to zero.

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